ISRAEL JOURNAL OF MATHEMATICS **166** (2008), 313–347
DOI: 10.1007/s11856-008-1033-y

FINITE 2-GROUPS WITH EXACTLY ONE NONMETACYCLIC MAXIMAL SUBGROUP

BY

Zvonimir Janko

Mathematical Institute, University of Heidelberg,
69120 Heidelberg, Germany
e-mail: janko@mathi.uni-heidelberg.de

ABSTRACT

We determine here the structure of the title groups. All such groups G will be given in terms of generators and relations, and many important subgroups of these groups will be described. Let $d(G)$ be the minimal number of generators of G. We have here $d(G) \leq 3$ and if $d(G) = 3$, then G' is elementary abelian of order at most 4.

Suppose $d(G) = 2$. Then G' is abelian of rank ≤ 2 and G/G' is abelian of type $(2, 2^m)$, $m \geq 2$. If G' has no cyclic subgroup of index 2, then $m = 2$. If G' is noncyclic and $G/\Phi(G')$ has no normal elementary abelian subgroup of order 8, then G' has a cyclic subgroup of index 2 and $m = 2$. But the most important result is that for all such groups (with $d(G) = 2$) we have $G = AB$, for suitable cyclic subgroups A and B.

Conversely, if $G = AB$ is a finite nonmetacyclic 2-group, where A and B are cyclic, then G has exactly one nonmetacyclic maximal subgroup. Hence, in this paper the nonmetacyclic 2-groups which are products of two cyclic subgroups are completely determined. This solves a long-standing problem studied from 1953 to 1956 by B. Huppert, N. Itô and A. Ohara. Note that if $G = AB$ is a finite p-group, $p > 2$, where A and B are cyclic, then G is necessarily metacyclic (Huppert [4]). Hence, we have solved here problem Nr. 776 from Berkovich [1].

Received June 27, 2006 and in revised form January 05, 2007

1. Introduction

Let G be a nonmetacyclic finite 2-group. If all maximal subgroups of G are metacyclic, then G is minimal nonmetacyclic and then $d(G) = 3$, $|G| \leq 2^5$, and there are exactly four such groups (see [3, Theorem 7.1]).

It is natural to ask what happens if all maximal subgroups except one are metacyclic. In that case the situation is essentially more complicated, since there exist many infinite classes of such finite 2-groups.

We determine here the structure of all finite 2-groups G which have exactly one nonmetacyclic maximal subgroup. All such groups G will be given in terms of generators and relations but we shall also describe many important subgroups of these groups. It is easy to see that we must have $d(G) \leq 3$.

If $d(G) = 3$, then the problem is simpler because in this case the group G has six metacyclic maximal subgroups. Such groups are given in Theorem 3.8 and we see that there are exactly five infinite classes of these groups. It is interesting to note that in all such groups the commutator subgroup G' is elementary abelian of order ≤ 4.

Now assume $d(G) = 2$, this is essentially more difficult. In this case we show that G/G' is abelian of type $(2, 2^m)$, $m \geq 2$, and $G' \neq \{1\}$ is abelian of rank ≤ 2. If G has a normal elementary abelian subgroup of order 8, then these groups are determined in Theorems 4.1 and 4.2. If G has no normal elementary abelian subgroup of order 8, then many properties of such groups are described in detail in Theorem 4.3. In fact, this theorem is a key result for further case-to-case investigations depending on the structure of G' and $G/\Phi(G')$. It is interesting to note that if G' is noncyclic but $G/\Phi(G')$ has no normal elementary abelian subgroup of order 8, then G' has a cyclic subgroup of index 2 and $m = 2$ (i.e., G/G' is abelian of type $(2, 4)$) and such groups are determined in Theorems 4.6 and 4.7, where we get an exceptional group of order 2^5 and two infinite classes. However, if G' has no cyclic subgroup of index 2, then $m = 2$, $Z(G)$ is elementary abelian of order ≤ 4 (Theorem 4.10) and all such groups are completely determined in Theorems 4.9, 4.11, 4.12, and 4.13 (where we get infinite classes of groups in each case). If G' is cyclic or if G' is noncyclic but G' has a cyclic subgroup of index 2 and $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8, then such groups are determined in Theorems 4.4 and 4.8. This exhausts all possibilities. The most impressive result is Corollary 4.5, where it is shown that in each case with $d(G) = 2$ such a group $G = AB$ is a product of two suitable cyclic subgroups A and B.

The converse of the last result is Theorem 5.1 which was also proved independently by Y. Berkovich. There it was proved that if $G = AB$ is a finite nonmetacyclic 2-group, where A and B are cyclic, then G has exactly one nonmetacyclic maximal subgroup and so all such groups have been completely determined in our previous theorems for $d(G) = 2$.

In each infinite class of 2-groups (given in terms of generators and relations) we have checked several smallest groups with a computer (coset enumeration program) and so we have proved that they exist. Actually, we have obtained faithful permutation representations for these groups.

The groups appearing in Theorems 4.7–4.13 depend on a number of parameters. It is clear that two groups that appear in different theorems are nonisomorphic, but different parameters in the same theorem could give isomorphic groups. The referee found some isomorphic groups for different choices of parameters in Theorem 4.7(a) and Theorem 4.8. Similar phenomena occurs in other theorems of Section 4. Also, the referee noticed that some groups in these theorems can be obtained as quotients of ones in other theorem. This means that the isomorphism problem is not solved and an investigation in that direction could inspire further research.

Finally, it is easily checked that all 2-groups given in our theorems have exactly one nonmetacyclic maximal subgroup.

We consider only finite p-groups and we use the standard notation. A 2-group H is said to be an L_3-group if H has a normal elementary abelian subgroup E of order 8 such that H/E is cyclic of order ≥ 4 and $\Omega_1(H) = E$. A metacyclic 2-group H is called "ordinary" metacyclic (with respect to A) if H has a cyclic normal subgroup A such that H/A is cyclic and H centralizes $A/\mho_2(A)$. A 2-group H is said to be a U_2-group (with respect to the kernel R) if H has a normal four-subgroup R such that H/R is of maximal class and if T/R is a cyclic subgroup of index 2 in H/R, then $\Omega_1(T) = R$. A 2-group H is "powerful" if $H' \leq \mho_2(H)$.

2. Known results

Our proofs are elementary but they are very involved and therefore we state here some known results which are used often in this paper.

Proposition 2.1 ([3, Lemma 1.1(l)]): *A p-group G is metacyclic if and only if* $\Omega_2(G)$ *is.*

Proposition 2.2 ([3, Lemma 1.1(n)]): *A 2-group G is metacyclic if and only if G and all maximal subgroups of G are generated by two elements.*

Proposition 2.3 ([3, Lemma 1.1(o)]): *A 2-group G is metacyclic if and only if the factor-group* $G/\mho_2(G)$ *is.*

Proposition 2.4 (O. Taussky, see [3, Lemma 1.1(s)]): *If G is a nonabelian 2-group with* $|G:G'|=4$, *then G is a dihedral, semidihedral or generalized quaternion. These three series of groups exhaust all 2-groups of maximal class.*

Proposition 2.5 (A. Mann, see [3, Lemma 1.1(u)]): *If U and V are distinct maximal subgroups in a p-group G, then* $|G':(U'V')|\le p$.

Proposition 2.6 (Burnside, see [3, Lemma 1.1(v)]): . *Let G be a p-group and M be a G-invariant subgroup of the Frattini subgroup* $\Phi(G)$. *If* $Z(M)$ *is cyclic, then so is M.*

Proposition 2.7 ([3, Lemma 1.1(z)]): . *If a metacyclic p-group G possesses a nonabelian subgroup of order* p^3, *then G is of maximal class.*

Proposition 2.8 (Rédei, see [3, Lemma 3.1]): *Let G be a minimal nonabelian p-group. Then* $G=\langle a,b\rangle$, $|G'|=p$, *and* $Z(G)=\Phi(G)$. *The group G is nonmetacyclic if and only if G′ is a maximal cyclic subgroup of G and in that case we have*

$$G=\langle a,b \mid a^{p^m}=b^{p^n}=c^p=1,\ [a,b]=c,\ [a,c]=[b,c]=1,$$
$$m\ge n\ge 1 \text{ and if } p=2, \text{ then } m>1\rangle,$$

where $|G|=p^{m+n+1}$. *If* $p=2$, *then* $\Omega_1(G)\cong E_8$, *where* E_{2^s} *denotes the elementary abelian group of order* 2^s.

Proposition 2.9 ([3, Lemma 3.2(a)]): *Let G be a p-group. If* $|G'|=p$ *and* $d(G)=2$, *then G is minimal nonabelian.*

Proposition 2.10 ([6]): *A two-generator 2-group is powerful if and only if it is ordinary metacyclic.*

Proposition 2.11 ([6]): *Let* $G=\langle a_1,a_2,\dots,a_n\rangle$ *be a powerful 2-group. Then* $G=\langle a_1\rangle\langle a_2\rangle\cdots\langle a_n\rangle$.

PROPOSITION 2.12 ([4, Satz 2]): *Let $G = \langle a\rangle\langle b\rangle$ be a p-group. Then each subgroup $\langle a^i\rangle$ of $\langle a\rangle$ is permutable with each subgroup $\langle b^j\rangle$ of $\langle b\rangle$. Also, we have*

$$\mho_1(G) = \Phi(G) = \langle a^p\rangle\langle b^p\rangle.$$

PROPOSITION 2.13 ([5, Theorem 1.2]): *Let G be a 2-group which does not have a normal elementary abelian subgroup of order 8. Suppose that G has (at least) two distinct normal four-subgroups U and V. Then $D = UV \cong D_8$ (a dihedral group of order 8) and $G = D * C$ (a central product) with $D \cap C = Z(D)$ and C is either cyclic or of maximal class distinct from D_8.*

PROPOSITION 2.14 ([2, Lemma 5]): *Let G be a nonabelian two-generator p-group. Then $G'/K_3(G)$ is cyclic (where $K_3(G) = [G, G']$). In particular, if R is a G-invariant subgroup of index p in G', then $R = \Phi(G')K_3(G)$ and so R is unique.*

PROPOSITION 2.15 ([2, Theorem 1]): *A nonabelian p-group G is metacyclic if and only if G/R is metacyclic for some G-invariant subgroup R of index p in G'.*

PROPOSITION 2.16 ([3, Theorem 11.2]): *Let G be a nonabelian group of order 2^m, $m \geq 5$, and exponent 2^{m-2}. Then one of the following holds:*

(a) *G is an L_3-group.*
(b) *G is a uniquely determined group of order 2^5 with $\Omega_2(G) \cong D_8 \times C_2$, where it turns out that G possesses a normal elementary abelian subgroup of order 8.*
(c) *G is a U_2-group and so G' is cyclic of index 8 in G.*
(d) *G is metacyclic.*
(e) *$G = QZ$, where $Q \cong Q_8$ is a normal subgroup of G, $Q \cap Z = Z(Q)$ and $Z = \langle b\rangle \cong C_{2^{m-2}}$, where b either centralizes Q or b induces on Q an involutory outer automorphism (in which case $m > 5$ and $\Phi(G) = G'\langle a^2\rangle$ is noncyclic). In any case, G' is cyclic of order ≤ 4.*
(f) *G is a uniquely determined group of order 2^5 with $\Omega_2(G) = \langle a, b\rangle \times \langle u\rangle$, where $Q = \langle a, b\rangle \cong Q_8$ and u is an involution. Set $\langle z\rangle = Z(Q)$. There is an element y of order 8 in G such that*

$$y^2 = ua, \quad u^y = uz, \quad a^y = a^{-1}, \quad b^y = bu.$$

Here $\Phi(G) = \langle y^2, u\rangle$ *is abelian of type* $(4,2)$ *and so* $d(G) = 2$, $G' = \langle u, z\rangle \cong E_4$, *and* $Z(G) = \langle z\rangle \cong C_2$.

Proposition 2.17 ([3, Theorem 4.1]): *Let* G *be a nonabelian 2-group with* $d(G) = 3$ *and suppose that each maximal subgroup of* G *is generated by two elements. If* G *is of class* 2, *then* $|G| \le 2^6$ *and we have one of the following possibilities:*

(a) G *is a minimal nonmetacyclic group of order* $\le 2^5$.
(b) G *is a unique special group of order* 2^5 *with* $E_4 \cong Z(G) < \Omega_1(G) \cong E_8$ *and all three maximal subgroups containing* $\Omega_1(G)$ *are nonmetacyclic minimal nonabelian.*
(c) G *is a unique special group of order* 2^6 *with* $E_8 \cong Z(G) = \Omega_1(G)$ *and all maximal subgroups are nonmetacyclic minimal nonabelian. Here* G *is isomorphic to a Sylow* 2*-subgroup of* $Sz(8)$.

Proposition 2.18 ([3, Theorem 4.2]): . *Let* G *be a nonabelian 2-group with* $d(G) = 3$ *and suppose that each maximal subgroup of* G *is generated with two elements. If* G *is of class* > 2, *then* $G/K_3(G)$ *is isomorphic to the group of order* 2^6 *of Proposition 2.17(c).*

Proposition 2.19 (P. Roquette [7] and Y. Berkovich [1, Lemma 1.4]): . *Let* N *be a normal subgroup of a* p*-group* G. *If* N *does not possess a* G*-invariant elementary abelian subgroup of order* p^2, *then* N *is either cyclic or* $p = 2$ *and* N *is of maximal class.*

Proposition 2.20 ([1, Part 1, Introduction, Lemma 4]): *Let* G *be an abelian* p*-group. If* Z *is a cyclic subgroup of* G *of maximal possible order, it is a direct factor of* G.

3. The case $d(G) = 3$

We assume in this section that G *is a 2-group with exactly one nonmetacyclic maximal subgroup* M *and* $d(G) = 3$.

Suppose at the moment that $d(M) = 2$ so that all maximal subgroups of G are two-generated. Obviously, M is nonabelian and so G is nonabelian. If G is of class 2, we may apply Proposition 2.17. It follows that either each maximal subgroup of G is metacyclic or G has more than one nonmetacyclic maximal subgroup. This is a contradiction and so G is of class > 2. In that case we may apply Proposition 2.18 which implies that all maximal subgroups of G are nonmetacyclic, a contradiction. Hence $d(M) > 2$ and considering $M \cap F$, where F is a metacyclic maximal subgroup of G, we get $d(M) = 3$. We have proved

Lemma 3.1: *We have* $d(M) = 3$.

Now we shall determine the structure of G/G'. Since $G' \leq \Phi(G)$, we have $d(G/G') = 3$ and we want to show that $G' < \Phi(G)$. For that purpose we study the structure of $\bar{G} = G/\Phi(M)$, where $M/\Phi(M) \cong E_8$ (Lemma 3.1) and $|\Phi(G) : \Phi(M)| = 2$. Here $\bar{M}$ is elementary abelian of order 8, $|\bar{G} : \bar{M}| = 2$, and $\Phi(\bar{G}) = \langle z \rangle$, where z is an involution in $\bar{M}$. There is an element $a \in \bar{G} - \bar{M}$ such that $a^2 = z$. Suppose that $\bar{G}$ is nonabelian so that $\bar{G}' = \langle z \rangle$ and $\langle a \rangle$ is normal in G. We have $|\bar{M} : C_{\bar{M}}(a)| = 2$ and let $t \in \bar{M} - C_{\bar{M}}(a)$ and $u \in C_{\bar{M}}(a) - \langle z \rangle$ so that $\langle a, t \rangle \cong D_8$ and $\bar{G} = \langle u \rangle \times \langle a, t \rangle \cong C_2 \times D_8$. But this group has two elementary abelian subgroups of order 8 (which are maximal in $\bar{G}$), a contradiction. Hence, $\bar{G}$ is abelian (of type $(2, 2, 4)$) and so $G' \leq \Phi(M)$, and G/G' is abelian of type $(2, 2, 2^m)$, $m > 1$. We have proved

Lemma 3.2: *The abelian group* G/G' *is of type* $(2, 2, 2^m)$, $m > 1$.

On the other hand, abelian groups of type $(2, 2, 2^m)$, $m > 1$, satisfy the assumptions of this section. Therefore, we assume in the sequel that $G' \neq \{1\}$.

Now suppose that G *has a normal elementary abelian subgroup* E *of order* 8. By Lemma 3.2 and our assumption that G is nonabelian, we have $|G| \geq 2^5$. Since G has exactly one nonmetacyclic maximal subgroup, G/E is cyclic of order ≥ 4. Let $a \in G - E$ so that $\langle a \rangle$ covers G/E. Then $\{1\} \neq G' = [E, \langle a \rangle] < E$ and $\Phi(G) = G'\langle a^2 \rangle$. But $d(G) = 3$ implies $|G'| = 2$ and so a induces on E an automorphism of order 2. In particular, a^2 centralizes E and so (since E_{16} cannot be a subgroup of G) $E \cap \langle a \rangle = \langle z \rangle$ is of order 2. Hence $\Phi(G) = \langle a^2 \rangle \geq \langle z \rangle$ and so $G' = \langle z \rangle$ and $o(a) = 2^n$, $n \geq 3$. We can choose $u, v \in E$ so that $E = \langle u, v, z \rangle$ with $u^a = u$ and $v^a = vz$. Then $\langle a, v \rangle \cong M_{2^{n+1}}$ and $G = \langle u \rangle \times \langle a, v \rangle \cong C_2 \times M_{2^{n+1}}$, where $M_{2^{n+1}} = \langle a, v \,|\, a^{2^n} = v^2 = 1, n \geq 3, [v, a] = a^{2^{n-1}} \rangle$. The structure of G is uniquely determined. We have proved

Lemma 3.3: *If* $G' \neq \{1\}$ *and* G *possesses a normal elementary abelian subgroup of order* 8, *then* $G \cong C_2 \times M_{2^{n+1}}$, $n \geq 3$.

In the sequel we also assume that G *has no normal elementary abelian subgroup of order* 8.

Suppose for a moment that $\Phi(G)$ is cyclic. Since $\Phi(G) = \mho_1(G)$, it follows that G has a cyclic subgroup of index 4. We may use Proposition 2.16. Since G has no normal E_8 the cases (a) and (b) of that proposition are ruled out. If G is a U_2-group (case (c)), then $|G/G'| = 8$ which contradicts Lemma 3.2. Since G is nonmetacyclic and $d(G) = 3$, we see that the only possibility is $G = Q * Z$, where $Q \cong Q_8$ and $Z \cong C_{2^n}$ with $Q \cap Z = Z(Q)$ (from case (e)). By Lemma 3.2, $|G/G'| \geq 2^4$ and so $n \geq 3$. We have $Z(G) = Z$, set $Q = \langle x, y \rangle$ and let v be an element of order 4 in Z so that $v^2 = x^2 = y^2 = z$, where $\langle z \rangle = Q \cap Z$. Then yv is an involution so that $D = \langle x, yv \rangle \cong D_8$ and $G = D * Z$ with $D \cap Z = Z(D)$. We see that G has in this case two distinct normal four-subgroups which are contained in D.

Conversely, assume that G possesses two distinct normal four-subgroups. By Proposition 2.13, $G = D * Z$ with $D \cong D_8$, $D \cap Z = Z(D)$, and Z is either cyclic or of maximal class. But in our case $d(G) = 3$ and so Z must be cyclic thus we have obtained the group of the previous paragraph. We have proved

Lemma 3.4: *Suppose that G is nonabelian and G does not have a normal elementary abelian subgroup of order 8. Then the following two assumptions are equivalent:*

(a) *$\Phi(G)$ is cyclic.*
(b) *G has two distinct normal four-subgroups.*

If G satisfies (a) or (b), then $G = Q * Z$ with $Q \cong Q_8$, $Z \cong C_{2^n}$, $n \geq 3$, and $Q \cap Z = Z(Q)$.

In the sequel we assume also that $\Phi(G)$ is noncyclic which is equivalent with the assumption that G has a unique normal four-subgroup W.

Since $\Phi(G)$ is metacyclic but noncyclic, it follows by a result of Burnside (Proposition 2.6) that $W = \Omega_1(Z(\Phi(G))) \cong E_4$. In particular, $|G| \geq 2^5$. Let X be a metacyclic maximal subgroup of G. Let $i \in X - W$ be an involution. Since i cannot centralize W (because X is metacyclic), it follows $\langle W, i \rangle \cong D_8$. By Proposition 2.7, X is of maximal class. This is a contradiction, since $|X| \geq 2^4$ and X possesses the normal four-subgroup W. We have proved the following result

Lemma 3.5: *Let G be nonabelian without a normal E_8 and having a unique normal four-subgroup W. If X is any metacyclic maximal subgroup of G, then $\Omega_1(X) = W$ and X is not of maximal class.*

Now assume in addition that $G' \cong C_{2^r}$, $r \geq 1$. By Lemma 3.2, $G = EF$ with normal subgroups E and F, where $E \cap F = G'$, $E/G' \cong E_4$, and $F/G' \cong C_{2^m}$, $m \geq 2$. Let $a \in F$ be such that $\langle a \rangle$ covers F/G'. Then, $\Phi(G) = G'\langle a^2 \rangle$ and $W = \Omega_1(Z(\Phi(G)))$ is a unique normal four-subgroup of G. Since $W \not\leq E$, E does not have a G-invariant four-subgroup. Because E is noncyclic, the Roquette's lemma (Proposition 2.19) implies that E is of maximal class with $|E| = 2^{r+2}$ and $E' = G' = \Phi(E)$. We note that $M = E\Phi(G) = E\langle a^2 \rangle$ is the unique nonmetacyclic maximal subgroup of G, since $E\langle a^{2^{m-1}} \rangle / G' \cong E_8$. Hence, each maximal subgroup of G containing F is metacyclic. Suppose that there is an involution $i \in E - G'$. Then $X = F\langle i \rangle$ is a metacyclic maximal subgroup of G with $\Omega_1(X) > W$, contrary to Lemma 3.5. Since there are no involutions in $E - G'$, we have that $E \cong Q_{2^{r+2}}$, $r \geq 1$, is generalized quaternion.

Suppose $r > 1$. Let y be an element of order 4 in $E - G'$ so that $y^2 \in \Omega_1(G')$ and $Y = F\langle y \rangle$ is a metacyclic maximal subgroup of G. Since $|G'| \geq 4$, there is an element v of order 4 in G' so that $\langle y, v \rangle \cong Q_8$ is a nonabelian subgroup of order 8 contained in Y. By Proposition 2.7, Y is of maximal class, contrary to Lemma 3.5.

We have proved that $r = 1$ and so $G' \cong C_2$ and $E \cong Q_8$. Since $\Phi(G)$ is noncyclic, $\langle a \rangle$ splits over G', and so $F = G' \times \langle a \rangle$ with $o(a) = 2^m$, $m \geq 2$. Suppose that a induces an outer automorphism on E. But then $[E, \langle a \rangle] \cong C_4$, a contradiction. Hence, a induces an inner automorphism on E which implies $G = E * C$ with $E \cap C = G'$ and $C/G' \cong C_{2^m}$. Since $\Phi(G) = \Phi(C)$ and $\Phi(G)$ is noncyclic, C splits over G'. We have proved

LEMMA 3.6: *Let G be nonabelian without a normal E_8 and having a unique normal four-subgroup. If G' is cyclic, then $G' \cong C_2$ and $G = Q \times Z$, where $Q \cong Q_8$ and $Z \cong C_{2^m}$, $m \geq 2$.*

From now on we assume that G' is noncyclic. We know from the above that this assumption implies that G has no normal E_8, $\Phi(G)$ is noncyclic, and G has a unique normal four-subgroup W.

For the start, we assume in addition that $G' \cong E_4$. Using Lemma 3.2, we have $G = EF$ with normal subgroups E and F, where $E \cap F = G'$, $E/G' \cong E_4$ and $F/G' \cong C_{2^m}$, $m \geq 2$. Let M be the maximal subgroup of G containing E so that $d(M) = 3$. Since $F \not\leq M$, each maximal subgroup of G containing F is metacyclic but not of maximal class (Lemma 3.5). In particular, $\Omega_1(F) = G'$.

Let $a \in F - M$. Then $\langle a \rangle$ covers F/G' and $\langle a \rangle \cap G' = \langle z \rangle \cong C_2$ so that $o(a) = 2^{m+1}$, $m \geq 2$, and $z = a^{2^m}$.

Since $F' \leq \langle a \rangle \cap G' = \langle z \rangle$, F is either abelian of type $(2^{m+1}, 2)$ or $F \cong M_{2^{m+2}}$. In any case, $\Phi(G) = G'\langle a^2 \rangle$ is abelian of type $(2^m, 2)$, $m \geq 2$.

Let i be an involution in $G - F$. Then $X = F\langle i \rangle$ is a metacyclic maximal subgroup of G with $\Omega_1(X) = G'$ (Lemma 3.5), a contradiction. We have proved that $\Omega_1(G) = G'$ and so G has only three involutions.

Let $x \in E - G'$ such that x does not centralize G'. Then $x^2 \in G'$ and so $\langle G', x \rangle \cong D_8$. But in that case there are involutions in $\langle G', x \rangle - G'$, a contradiction. We have proved that $G' \leq Z(E)$.

Set $v = a^{2^{m-1}}$ so that $o(v) = 4$ and $v^2 = z$. Since $v \in \Phi(G)$, v centralizes G'. If X is any maximal subgroup of G containing F, then X is metacyclic and therefore X' is cyclic of order at most 2 (since $X' \leq G' \cong E_4$). In particular, X is of class ≤ 2. For any $x \in E - G'$, $F\langle x \rangle$ is a maximal subgroup of G containing F and so $[x, a^2] = [x, a]^2 = 1$ which gives $[E, a^2] = 1$. If for some $y \in E - G'$, $y^2 = z$, then

$$(yv)^2 = y^2 v^2 [v, y] = zz = 1$$

and so yv is an involution in $G - E$, a contradiction. We have proved that z is not a square in E. In particular, E is nonabelian (since E has exactly three involutions and $\exp(E) = 4$).

Since $z \in \Phi(\Phi(G)) = \langle a^4 \rangle$, we have $z \in Z(G)$. Take an $x \in E - G'$ so that $x^2 \in G' - \langle z \rangle$ and $F\langle x \rangle$ is of class ≤ 2. This gives $[x^2, a] = [x, a]^2 = 1$ and $C_G(x^2) \geq \langle E, a \rangle = G$. We have proved that $G' \leq Z(G)$, G is of class 2 and F is abelian of type $(2^{m+1}, 2)$.

We have $\Phi(G) = G'\langle a^2 \rangle \leq Z(G)$. If $Z(G) > \Phi(G)$, then $G/Z(G) \cong E_4$ and each (of the three) maximal subgroups of G containing $Z(G)$ is abelian. In that case a result of A. Mann (Proposition 2.5) implies $|G'| \leq 2$, a contradiction. We have proved $Z(G) = \Phi(G)$.

If E is minimal nonabelian, then the fact that $\Omega_1(E) = G' \cong E_4$ implies that E is metacyclic of exponent 4 (see Proposition 2.8) and so there are elements x and y of order 4 in $E - G'$ so that

$$E = \langle x, y \mid x^4 = y^4 = 1,\ x^y = x^{-1} \rangle,$$

where $E' = \langle x^2 \rangle$, $x^2 \neq y^2$, $x^2 \in G' - \langle z \rangle$, $y^2 \in G' - \langle z \rangle$, and $x^2 y^2 = z$ because z is not a square in E.

If E is not minimal nonabelian, then the fact that E has only three involutions and z is not a square in E implies $E = Q \times \langle z \rangle$ with $Q \cong Q_8$.

Suppose that E is minimal nonabelian given above. We note that $v = a^{2^{m-1}}$ centralizes E and $v^2 = z$. Replace y with $y' = vy$ so that

$$(y')^2 = (vy)^2 = v^2y^2 = zy^2 = x^2 \quad \text{and} \quad x^{y'} = x^{vy} = x^y = x^{-1}$$

and, therefore, $\langle x, y' \rangle = Q \cong Q_8$ and $E^* = Q \times \langle z \rangle$ is another complement of F modulo G'. Indeed, note that $E^* > G' = \Omega_1(G)$, $E^*/G' \cong E_4$, E^* is normal in G and $E^* \cap F = G'$. Hence, replacing E with E^*, if necessary, we may assume from the start that E is not minimal nonabelian and so $E = Q \times \langle z \rangle$, $Q \cong Q_8$, and setting $\langle u \rangle = Q' = Z(Q)$, we have $G' = \langle u \rangle \times \langle z \rangle \cong E_4$, $M = Q \times \langle a^2 \rangle \cong Q_8 \times C_{2^m}$, and $\Phi(M) = \langle u \rangle \times \langle a^4 \rangle$, where $\langle a^4 \rangle \geq \langle z \rangle$.

Set $Q = \langle x, y \rangle$ so that $x^2 = y^2 = [x, y] = u$. We have $G = Q\langle a \rangle$ with $Q \cong Q_8$, $Q \cap \langle a \rangle = \{1\}$, $o(a) = 2^{m+1}$, $m \geq 2$, and a^2 centralizes Q.

Let $l \in G - M$ so that $l = a^i q$, where i is an odd integer and $q \in Q$. Then (noting that G is of class 2), we get $l^2 = a^{2i}q^2[q, a^i]$,where $q^2[q, a^i] \in G'$. Hence, $o(l^2) = 2^m$ and $l^4 = (a^4)^i$ and so $\langle l^4 \rangle \geq \langle z \rangle$. Hence, each element $l \in G - M$ is of order 2^{m+1} and $\langle l \rangle \geq \langle z \rangle$.

The element a does not normalize Q (otherwise, $G' \leq Q$ and G' would be cyclic). Since $|E : Q| = 2$ and E is normal in G, we have $Q \cap Q^a = \langle x \rangle \cong C_4$ and $(Q \cap Q^a)^a = Q^a \cap Q^{a^2} = Q^a \cap Q$ (since a^2 centralizes Q) so that $\langle x \rangle^a = \langle x \rangle$. If $x^a = x^{-1}$, then we replace a with $a' = ay$, where $y \in Q - \langle x \rangle$. We get $x^{a'} = x^{ay} = (x^{-1})^y = x$ and so a' centralizes x, $o(a') = 2^{m+1}$ and $\langle a' \rangle \geq \langle z \rangle$. Hence we may assume from the start that $x^a = x$ and the maximal subgroup $A = \langle x \rangle \times \langle a \rangle$ is abelian of type $(4, 2^{m+1})$. By a result of Mann (Proposition 2.5), A is a unique abelian maximal subgroup of G. If $y \in Q - \langle x \rangle$, then $[y, a] \in G' - \langle u \rangle$ (otherwise Q would be normal in G). Suppose that $[y, a] = uz$. Then replace a with $a^* = ax$ (noting that a^* centralizes x, $o(a^*) = 2^{m+1}$, and $\langle a^* \rangle \geq \langle z \rangle$), we get

$$[y, a^*] = [y, ax] = [y, a][y, x] = uz \cdot u = z.$$

Thus, we may assume from the start that $[y, a] = z$. We have proved

Lemma 3.7: *Suppose that G' is a four-group. Then $G = Q\langle a \rangle$, where $Q = \langle x, y \rangle \cong Q_8$, $o(a) = 2^n$, $n \geq 3$, $Q \cap \langle a \rangle = \{1\}$, a^2 centralizes Q, $[a, x] = 1$ and $[a, y] = a^{2^{n-1}} = z$. Here $G' = \langle u, z \rangle \cong E_4$, where $\langle u \rangle = Z(Q)$,*

$\Phi(G) = Z(G) = \langle a^2 \rangle \times \langle u \rangle \cong C_{2^{n-1}} \times C_2$, and $M = Q \times \langle a^2 \rangle$ is a unique nonmetacyclic maximal subgroup of G.

In the rest of this section we assume that G' is noncyclic but G' is not isomorphic to a four-group. Since $G' \leq \Phi(G)$, G' is metacyclic and so $G'/\Phi(G') \cong E_4$ and $\Phi(G') \neq \{1\}$. Let R be a G-invariant subgroup of index 2 in $\Phi(G')$. We want to study the structure of G/R and so we may assume that $R = \{1\}$. In that case $\Phi(G') \cong C_2$ and a result of Burnside (Proposition 2.6) implies that G' is abelian of type $(4, 2)$. Here $W = \Omega_1(G')$ is a unique normal four-subgroup of G and $\langle z \rangle = \mho_1(G') \leq Z(G)$. By Lemma 3.2, $G = EF$ with normal subgroups E and F, where $E \cap F = G'$, $E/G' \cong E_4$ and $F/G' \cong C_{2^m}$, $m \geq 2$. Let $a \in F$ be such that $\langle a \rangle$ covers F/G' and $\Phi(G) = G'\langle a^2 \rangle$. Also, $M = E\langle a^2 \rangle$ is the unique nonmetacyclic maximal subgroup of G and so any proper subgroup of G which is not contained in M is metacyclic.

Let i be an involution in G and let X be a metacyclic maximal subgroup of G containing $F\langle i \rangle$. By lemma 3.5, $\Omega_1(X) = W$ and so $i \in W$. We have proved that $\Omega_1(G) = W$.

Suppose that $W \leq Z(G)$. In this case, take an involution $s \in W - \langle z \rangle$ and consider the group $G/\langle s \rangle$. We have $(G/\langle s \rangle)' \cong C_4$, which contradicts our previous results (which shows that a cyclic commutator group is of order at most 2). Hence $W \not\leq Z(G)$, so that $C_G(W)$ is a maximal subgroup of G and $\Omega_1(Z(G)) = \langle z \rangle$, which implies that $Z(G)$ is cyclic.

Let v be an element of order 4 in G' and let $u \in W - \langle z \rangle$. Then $v^2 = z$ and the set $\{\langle v \rangle, \langle vu \rangle\}$ is the set of cyclic subgroups of order 4 in G'. Suppose that $\langle v \rangle$ is not normal in G (and then also $\langle vu \rangle$ is not normal in G). Let $\{X_1, X_2, X_3\}$ be the set of maximal subgroups of G containing F. Since X_i is metacyclic, X_i' is cyclic for each $i = 1, 2, 3$. By our assumption (and noting that $W \not\leq Z(G)$), we get $X_i' \leq \langle z \rangle$. However, by a result of A. Mann (Proposition 2.5), this gives a contradiction.

We have proved that $\langle v \rangle$ and $\langle vu \rangle$ are normal subgroups in G. This implies that $\Phi(G) \leq C_G(v) \cap C_G(vu)$ and so $G' \leq Z(\Phi(G))$ because $G' = \langle v, vu \rangle$. But $\Phi(G)/G'$ is cyclic and so $\Phi(G)$ is abelian. In particular, a^2 centralizes G'.

We want to determine the subgroup $\langle a^{2^m} \rangle \leq G'$. If $a^{2^m} = 1$, then $a^{2^{m-1}}$ is an involution in $F - G'$, contrary to our result that $\Omega_1(G) = W \leq G'$. Hence $a^{2^m} \neq 1$. If $a^{2^m} = z$, then $a^{2^{m-1}} \in \Phi(G) - G'$, $o(a^{2^{m-1}}) = 4$, and $a^{2^{m-1}}v$ is an involution in $\Phi(G) - G'$, a contradiction. Suppose that $a^{2^m} = u \in W - \langle z \rangle$ and so $F = \langle a\rangle\langle v\rangle$, $\langle a\rangle \cap \langle v\rangle = \{1\}$ and a normalizes $\langle v\rangle$ (since $\langle v\rangle$ is normal in G). We get $v^a = vz^\epsilon$, $\epsilon = 0, 1$ and so $F' \in \langle z\rangle$ which implies that $\mho_2(F) = \langle a^4\rangle \geq \langle u\rangle$. It follows that $\langle u\rangle$ is a characteristic subgroup in F and so $u \in Z(G)$, a contradiction. Hence, replacing $\langle v\rangle$ with $\langle vu\rangle$ and v with v^{-1}, if necessary, we may assume that $a^{2^m} = v$. Thus, $o(a) = 2^{m+2}$ and so $\langle a\rangle$ is a cyclic subgroup of index 2 in F. Since F is not of maximal class ($W \cong E_4$ is normal in F and $|F| \geq 2^5$), F is either abelian or $F \cong M_{2^{m+3}}$. In any case, $F' \leq \langle z\rangle$ and $a^2 \in Z(F)$.

We claim that $v \in Z(G)$. If q is an element in E, then $[q, a] \in G'$ and so

$$[q, a^2] = [q, a][q, a]^a = [q, a][q, a]z^\epsilon = [q, a]^2 z^\epsilon = z^\eta, \quad \epsilon, \eta = 0, 1,$$

since $[q, a]^2 \in \mho_1(G') = \langle z\rangle$. This gives

$$[q, a^4] = [q, a^2][q, a^2]^{a^2} = z^\eta (z^\eta)^{a^2} = (z^\eta)^2 = 1.$$

But $a^{2^m} = v$, $m \geq 2$, and so $\langle a^4\rangle \geq \langle v\rangle$ which implies that v centralizes E. It follows $C_G(v) \geq \langle E, a\rangle = G$ and we are done.

Now we use Lemma 3.7 for the group $G/\langle z\rangle$ since $(G/\langle z\rangle)' = G'/\langle z\rangle \cong E_4$. It follows that $G/\langle z\rangle$ possesses a quaternion subgroup $\tilde{Q}/\langle z\rangle \cong Q_8$. Suppose that $v \in \tilde{Q}$. Then $\Phi(\tilde{Q}) = \langle v\rangle$ and $\tilde{Q}$ possesses a cyclic subgroup of index 2. But such groups cannot have a proper homomorphic image $\tilde{Q}/\langle z\rangle$ isomorphic to Q_8. Hence, $v \notin \tilde{Q}$ and so $\tilde{Q} \cap \langle v\rangle = \langle z\rangle$. If $|\tilde{Q}'| = 4$, then a result of O. Taussky (Proposition 2.4) implies that $\tilde{Q}$ is of maximal class. This is again a contradiction since $\tilde{Q}/\langle z\rangle \cong Q_8$. Hence $\tilde{Q}' = \langle u\rangle$ is of order 2 and $u \neq z$ since $\tilde{Q}/\langle z\rangle$ is nonabelian. We get $\langle u, v\rangle = G'$ and therefore $E^* = \tilde{Q} * \langle v\rangle$ is normal in G. But $(E^*)' = \tilde{Q}' = \langle u\rangle$ is a characteristic subgroup of E^* and so $u \in Z(G)$. This gives $W = \langle u, z\rangle \leq Z(G)$ and this is our final contradiction. We have proved that such a group G does not exist. We conclude with the following result which sums up all results of this section.

Theorem 3.8: *Let G be a 2-group which possesses exactly one nonmetacyclic maximal subgroup M. Then $d(G) \leq 3$ and we assume here $d(G) = 3$. In that case $d(M) = 3$, G/G' is abelian of type $(2, 2, 2^m)$, $m \geq 2$, G' is elementary abelian of order ≤ 4, and we have exactly the following five possibilities:*

(a) *G is abelian of type $(2, 2, 2^m)$, $m \geq 2$.*
(b) *$G \cong C_2 \times M_{2^{n+1}}$, $n \geq 3$, where $M_{2^{n+1}} = \langle a, v \mid a^{2^n} = v^2 = 1, [v, a] = a^{2^{n-1}}\rangle$.*

(c) $G = Q * Z$, *where* $Q \cong Q_8$, $Z \cong C_{2^n}$, $n \geq 3$, *and* $Q \cap Z = Z(Q)$.

(d) $G = Q \times Z$, *where* $Q \cong Q_8$, $Z \cong C_{2^n}$, $n \geq 2$.

(e) $G = QZ$, *where* $Q = \langle x, y \rangle \cong Q_8$, $Z = \langle a \rangle \cong C_{2^n}$, $n \geq 3$, $Q \cap Z = \{1\}$, a^2 *centralizes* Q, $[a, x] = 1$, *and* $[a, y] = a^{2^{n-1}} = z$. *Setting* $Z(Q) = \langle u \rangle$, *we have here* $G' = \langle u, z \rangle \cong E_4$, $\Phi(G) = Z(G) = \langle a^2 \rangle \times \langle u \rangle \cong C_{2^{n-1}} \times C_2$, *and* $M = Q \times \langle a^2 \rangle$.

Conversely, it is easily checked that all groups G *given in (a) to (e) have exactly one nonmetacyclic maximal subgroup and* $d(G) = 3$.

4. The case $d(G) = 2$

We assume in this section that G is a 2-group with exactly one nonmetacyclic maximal subgroup M and $d(G) = 2$. By Proposition 2.2 follows at once that $d(M) = 3$, G is nonmetacyclic and so $G' \neq \{1\}$.

First we treat the easy case $|G'| = 2$. By Proposition 2.9, G is minimal nonabelian. By Proposition 2.8, we have

$$G = \langle a, b \mid a^{2^m} = b^{2^n} = c^2 = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1, m \geq n \geq 1,\ m \geq 2 \rangle,$$

where $|G| = 2^{m+n+1}$, $\Omega_1(G) = \langle a^{2^{m-1}}, b^{2^{n-1}}, c \rangle \cong E_8$, and $G/\Omega_1(G) \cong C_{2^{m-1}} \times C_{2^{n-1}}$. Since there is only one maximal subgroup of G containing $\Omega_1(G)$, $G/\Omega_1(G)$ must be cyclic and this implies $n = 1$ so that G/G' is abelian of type $(2^m, 2)$, $m \geq 2$. We have $|G : \langle a \rangle| = 4$, $(ab)^2 = a^2 c$, $(ab)^4 = a^4$, and so $|\langle ab \rangle : (\langle ab \rangle \cap \langle a \rangle)| = 4$ which gives (by the product formula) $G = \langle a \rangle \langle ab \rangle$. Also, $\langle c \rangle = G'$ is a maximal cyclic subgroup of G. We have proved

THEOREM 4.1: *Let* G *be a 2-group with exactly one nonmetacyclic maximal subgroup and* $d(G) = 2$. *If* $|G'| = 2$, *then*

$$G = \langle a, b \mid a^{2^m} = b^2 = c^2 = 1,\ m \geq 2,\ [a, b] = c,\ [a, c] = [b, c] = 1 \rangle,$$

which is a nonmetacyclic minimal nonabelian group with G/G' *being abelian of type* $(2^m, 2)$, $m \geq 2$, $G' = \langle c \rangle$ *is a maximal cyclic subgroup of* G, $G = \langle a \rangle \langle ab \rangle$, *and* $\Omega_1(G) \cong E_8$ *so that* G *has a normal elementary abelian subgroup of order* 8.

Now assume that G has a normal elementary abelian subgroup E of order 8 but $|G'| > 2$. Then $G/E \neq \{1\}$ must be cyclic and since $G' < E$, we have $G' \cong E_4$. Let $a \in G - E$ be such that $\langle a \rangle$ covers G/E. Since $G' = [E, \langle a \rangle]$, a induces on E an automorphism of order 4 which implies $|G/E| \geq 4$. We have $\Phi(G) = G'\langle a^2\rangle$ and so $E \cap \langle a\rangle \leq G'$ (noting that $|G : \Phi(G)| = 4$). The maximal subgroup $M = E\langle a^2\rangle$ is nonmetacyclic and so the maximal subgroup $X = G'\langle a\rangle$ is metacyclic (of order $\geq 2^4$) with a normal four-subgroup G'. This implies that X is not of maximal class. If i is an involution in $X - G'$, then i cannot centralize G' (since X is metacyclic). But in that case $G'\langle i\rangle \cong D_8$ and so, by Proposition 2.7, X is of maximal class, a contradiction. Hence $\Omega_1(X) = G'$ and so $G' \cap \langle a\rangle = \langle z\rangle \cong C_2$. Let $v \in E - G'$ so that $[v, a] = u \in G' - \langle z\rangle$ and $[u, a] = z$. This gives $v^a = vu$, $u^a = uz$, $\langle z\rangle \leq Z(G)$, $o(a) = 2^n$, $n \geq 3$, and $a^{2^{n-1}} = z$. The structure of G is uniquely determined. We compute

$$(av)^2 = avav = a^2v^av = a^2(vu)v = a^2u \text{ and } (av)^4 = (a^2u)^2 = a^4.$$

Thus, $\langle av\rangle \cap \langle a\rangle = \langle a^4\rangle$ and since $|G : \langle a\rangle| = 4$ and $|\langle av\rangle : (\langle av\rangle \cap \langle a\rangle)| = 4$, we get $\langle av\rangle\langle a\rangle = G$. If $n \geq 4$, then $C_G(E) > E$ and, therefore, $\Omega_1(G) = E$. If $n = 3$, then $C_G(E) = E$ and $\Omega_1(G) = E\langle a^2\rangle = \langle u\rangle \times \langle v, a^2\rangle \cong C_2 \times D_8$. We have proved:

Theorem 4.2: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Suppose that G has a normal elementary abelian subgroup E of order 8 and $|G'| > 2$. Then $G' \cong E_4$ and we have $G = EZ$, $Z = \langle a\rangle$ is of order 2^n, $n \geq 3$, $E \cap Z = \langle z\rangle \cong C_2$, $z = a^{2^{n-1}}$, and setting $E = \langle u, v, z\rangle$, we have $u^a = uz$, $v^a = vu$. We have $G' = \langle u, z\rangle \cong E_4$, $Z(G) = \langle a^4\rangle \cong C_{2^{n-2}}$, $\Phi(G) = \langle u\rangle \times \langle a^2\rangle \cong C_2 \times C_{2^{n-1}}$, and $G = \langle av\rangle\langle a\rangle$. If $n > 3$, then $\Omega_1(G) = E$ and if $n = 3$, then $\Omega_1(G) = E\langle a^2\rangle \cong C_2 \times D_8$.*

In the rest of this section we assume that G has no normal elementary abelian subgroup of order 8. We prove the following key result which will be used (with the introduced notation and with all details) in the rest of this section.

Theorem 4.3: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume in addition that G has no normal elementary abelian subgroup of order 8. Then the following hold*

(a) *$|G'| > 2$ and $|G| \geq 2^5$.*

(b) *G has exactly one normal four-subgroup $W = \Omega_1(Z(\Phi(G)))$.*

(c) *For each metacyclic maximal subgroup X of G, $\Omega_1(X) = W$.*

(d) *Let R be a G-invariant subgroup of index 2 in G'. Then R is unique and G/R is isomorphic to a group of Theorem 4.1. In particular, G/G' is abelian of type $(2^m, 2)$, $m > 1$, $\Omega_1(G/R) \cong E_8$, and if y is any element in G such* *that $y^2 \in G'$, then $y^2 \in R$. Also, each proper characteristic subgroup of G' is contained in R.*

(e) *G' is abelian of rank ≤ 2.*

(f) *There are normal subgroups E and F of G such that $G = EF$, $E \cap F = G'$, $F/G' \cong C_{2^m}$, $m \geq 2$, $E/G' \cong C_2$, and there is an element $x \in E - G'$ of order ≤ 4 and we fix such an element x. Let $a \in F - G'$ be such that $\langle a \rangle$ covers F/G'. Then $\Phi(G) = G'\langle a^2 \rangle$, $\Omega_1(G/R) = (E\langle a^{2^{m-1}} \rangle)/R \cong E_8$, $M = E\langle a^2 \rangle$ is the unique nonmetacyclic maximal subgroup of G, $F = G'\langle a \rangle$ and $F_1 = G'\langle ax \rangle$ are two distinct metacyclic maximal subgroups of G, and $F'F_1' = R$. We have $G = \langle a, x \rangle$ and $v = [a, x] \in G' - R$.*

(g) *Assuming in addition that G' is noncyclic, we have the following properties:*

(g1) *All elements in $G' - R$ are of order $2^e = \exp(G')$. In particular, $o(v) = 2^e$.*

(g2) *If R is cyclic, then $|R| = 2$ and $G' \cong E_4$.*

(g3) *We have $G'/\langle v \rangle \cong R/\langle v^2 \rangle$ is cyclic of order $\leq 2^e$ and if y is any element in $R - \Phi(G')$, then $\langle y \rangle$ covers $R/\langle v^2 \rangle$ and $\langle v^2 \rangle$ has a cyclic complement of order $\leq 2^e$ in R.*

(g4) *If $\exp(R) = \exp(G') = 2^e$, then $G' \cong C_{2^e} \times C_{2^e}$ is homocyclic of rank 2 and if $\exp(R) < \exp(G') = 2^e$, then $\exp(R) = 2^{e-1}$.*

(g5) *We have $a^{2^m} \in R - \Phi(G')$ and $(ax)^{2^m} \in R - \Phi(G')$.*

(g6) *If $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8, then $\Phi(G') \neq \{1\}$ and we may assume that $E/\Phi(G') \cong E_8$ so that our fixed element $x \in E - G'$ with $o(x) \leq 4$ has in this case the additional property $x^2 \in \Phi(G')$.*

(g7) *We have $x^2 \in W \leq G'$ and if $G' \not\cong E_4$, then $W \leq R$.*

(g8) *We have $v^x = v^{-1}z^\epsilon$, where $\epsilon = 0, 1$ and $\epsilon = 1$ if and only if $x^2 \in W - Z(G)$ in which case $W \not\leq Z(G)$ and $\langle z \rangle = E \cap Z(G)$.*

(g9) *We have $F = \langle a \rangle\langle v \rangle$, $F_1 = \langle ax \rangle\langle v \rangle$, and $\Phi(G) = \langle a^2 \rangle\langle v \rangle$.*

(g10) *Setting $b = [v, a]$ (which is equivalent with $v^a = vb$) and $b_1 = [v, ax]$, we have $F' = \langle b \rangle$, $F_1' = \langle b_1 \rangle$ with $b, b_1 \in R - \Phi(G')$, $\langle b \rangle\langle b_1 \rangle = R$, $o(b) = \exp(R)$, and $b_1 = v^{-2}z^\epsilon b^{-1}$.*

(g11) *We have $b^x = b^{-1}$, $b_1^x = b_1^{-1}$ so that x inverts each element in R. Also, $b_1^{-1}b_1^a = (bb^a)^{-1} \in \langle b \rangle \cap \langle b_1 \rangle$.*

(g12) *We have $\Phi(G)' = \langle bb^a \rangle$ and $\Phi(G)$ is powerful.*

(g13) *We have $G = \langle ax \rangle\langle a \rangle$ and so G is a product of two cyclic subgroups.*

Proof. Since G has no normal E_8, we get $|G'| > 2$ (Theorem 4.1) and $|G| \geq 2^5$ because $|G/G'| \geq 2^3$ (O. Taussky). If G has two distinct normal four-subgroups, then Proposition 2.13 implies that $d(G) > 2$, a contradiction. Hence, by Proposition 2.19, G possesses exactly one normal four-subgroup W.

Let X be a metacyclic maximal subgroup of G so that $|X : \Phi(G)| = 2$. If $\Phi(G)$ is cyclic, then G has a cyclic subgroup of index 2, a contradiction. Hence $\Phi(G)$ is noncyclic and a result of Burnside (Proposition 2.6) implies that $Z(\Phi(G))$ is noncyclic. This gives $W = \Omega_1(Z(\Phi(G)))$, noting that $\Phi(G)$ is metacyclic. Since $|X| \geq 2^4$ and X has a normal four-subgroup, X is not of maximal class. Let i be an involution in $X - W$. Since X is metacyclic, i cannot centralize W. It follows $\langle W, i\rangle \cong D_8$ and then Proposition 2.7 implies that X is of maximal class, a contradiction. Hence, we have $\Omega_1(X) = W$ for each metacyclic maximal subgroup X of G.

Let R be a G-invariant subgroup of index 2 in G'. By Proposition 2.14, $R = \Phi(G')K_3(G)$ and so such a subgroup R is unique. Since G is nonmetacyclic, $\bar{G} = G/R$ is also nonmetacyclic (Proposition 2.15). If X is a metacyclic maximal subgroup of G, then $\bar{X}$ (bar convention) is metacyclic. If M is the unique nonmetacyclic maximal subgroup of G, then $\bar{M}$ is also nonmetacyclic (othervise, Proposition 2.2 would imply that $\bar{G}$ is metacyclic). Since $|\bar{G}'| = 2$, $\bar{G}$ must be isomorphic to a group of Theorem 4.1. In particular, G/G' is abelian of type $(2^m, 2)$, $m > 1$, $\Omega_1(G/R) \cong E_8$, and if y is any element in G such that $y^2 \in G'$, then $y^2 \in R$. The uniqueness of R also implies that each proper characteristic subgroup of G' is contained in R.

Let $X_1 \neq X_2$ be two metacyclic maximal subgroups of G so that X_1' and X_2' are cyclic normal subgroups of G. Since G/R is minimal nonabelian, we have $X_1'X_2' \leq R$. By a result of A. Mann (Proposition 2.5), we get $R = X_1'X_2'$. On the other hand, $G/C_G(X_1')$ and $G/C_G(X_2')$ are abelian groups and so G' centralizes $X_1'X_2' = R$. But $|G' : R| = 2$ and, therefore, G' is abelian (of rank ≤ 2).

Now we use the structure of G/R. There are normal subgroups E and F of G such that $G = EF$, $E \cap F = G'$, $F/G' \cong C_{2^m}$, $m \geq 2$, and $E/G' \cong C_2$. Let $a \in F - G'$ be such that $\langle a\rangle$ covers F/G'. Then $\Phi(G) = G'\langle a^2\rangle$ and $\Omega_1(G/R) = S/R \cong E_8$, where $S = E\langle a^{2^{m-1}}\rangle$ (because $E/R \cong E_4$ and $a^{2^m} \in R$). It follows that $M = E\langle a^2\rangle$ is the unique nonmetacyclic maximal subgroup of G (noting that already S is nonmetacyclic). Let x be any element in $E - G'$ so that $G = \langle a, x\rangle$, $F = G'\langle a\rangle$ and $F_1 = G'\langle ax\rangle$ are two metacyclic maximal subgroups of G, where we use the facts that $\langle ax\rangle$ also covers $G/E \cong C_{2^m}$ and $(ax)^2 \in \Phi(G)$. Set $S_0 = G'\langle a^{2^{m-1}}\rangle$ and $E_1 = G'\langle a^{2^{m-1}}x\rangle$ so that S_0 is a metacyclic maximal subgroup of S, E_1/G' is another complement of F/G' in G/G', and $S - S_0 = (E - G') \cup (E_1 - G')$. By Proposition 2.1, $\Omega_2(S) \nleq S_0$ and so there are elements of order ≤ 4 in $S - S_0$. Interchanging E and E_1, if necessary, we may assume from the start that there is an element $x \in E - G'$ with $o(x) \leq 4$ and we choose and fix such an element x. We set $v = [a, x]$ so that $v \in G' - R$. Indeed, we have $G = \langle a, x\rangle$ and so if $v \in R$, then G/R would be abelian.

In what follows we assume that G' is noncyclic so that $\Phi(G') < R$ and $G'/\Phi(G') \cong E_4$. Let $2^e = \exp(G')$, $e \geq 1$, be the exponent of G'. If there is an element in $G' - R$ of order $< 2^e$ (in which case $e > 1$), then $\Omega_{e-1}(G')$ is a proper characteristic subgroup of G' which is not contained in R, a contradiction. Hence, all elements in $G' - R$ are of order 2^e. In particular, the element $v \in G' - R$ of the previous paragraph is of order 2^e. By Proposition 2.20, $\langle v\rangle$ has a cyclic complement $\langle s\rangle$ of order $\leq 2^e$ (noting that G' is of rank 2). Hence, $G'/\langle v\rangle \cong R/\langle v^2\rangle$ is cyclic of order $\leq 2^e$. Since $v^2 \in \Phi(G')$, we have $\Phi(G') = \langle s^2\rangle \times \langle v^2\rangle$ and if y is any element in $R - \Phi(G')$, then $\langle y\rangle$ covers $R/\langle v^2\rangle$.

Suppose that R is cyclic of order > 2. Since G' is noncyclic, there is an involution in $G' - R$, contrary to the fact that all elements in $G' - R$ are of order $2^e = \exp(G')$. Hence, if R is cyclic, then $|R| = 2$ and $G' \cong E_4$.

Suppose that $\exp(R) = \exp(G') = 2^e$, $e > 1$. Let y be an element of order 2^e in R. Suppose also that $\langle y\rangle \cap \langle v^2\rangle \neq \{1\}$. Then we have

$$|\langle y\rangle : (\langle y\rangle \cap \langle v\rangle)| = |\langle v\rangle : (\langle y\rangle \cap \langle v\rangle)| = 2^{e'},\ e' < e,$$

and so there is an element y' of order 2^e in $\langle y\rangle$ such that $(y')^{2^{e'}} = v^{-2^{e'}}$. But then $(y'v)^{2^{e'}} = 1$ and $y'v \in G' - R$, a contradiction. Thus, $\langle y\rangle$ splits over $\langle v^2\rangle$, $R = \langle y\rangle \times \langle v^2\rangle$, and so $G' \cong C_{2^e} \times C_{2^e}$ is homocyclic of rank 2. But if G' is not homocyclic, then $\exp(R) = 2^{e-1}$ and so (by Proposition 2.20) $\langle v^2\rangle$ has a cyclic complement in R. It follows that in any case $\langle v^2\rangle$ has a cyclic complement in R.

Since $a^{2^m} \in G'$, we know that $a^{2^m} \in R$. Suppose that $a^{2^m} \in \Phi(G')$. We look at $F/\Phi(G') = \bar{F}$ so that $\bar{G}' = G'/\Phi(G') \cong E_4$ is a normal four-subgroup of the metacyclic group $\bar{F}$ (of order $\geq 2^4$) and so $\bar{F}$ is not of maximal class. But $\overline{a^{2^{m-1}}}$ is an involution in $\bar{F} - \bar{G}'$ and $\overline{a^{2^{m-1}}}$ cannot centralize $\bar{G}'$. It follows that $\langle \bar{G}', \overline{a^{2^{m-1}}}\rangle \cong D_8$ and this is a contradiction (by Proposition 2.7). We have proved that $a^{2^m} \in R - \Phi(G')$. With the same argument (working in $\bar{F}_1 = F_1/\Phi(G')$), we get $(ax)^{2^m} \in R - \Phi(G')$.

Assume for a moment that $G/\Phi(G')$ has a normal elementary abelian subgroup $S^*/\Phi(G')$ of order 8 so that $S^* < S$ and $|S : S^*| = 2$ (where $S/R = \Omega_1(G/R) \cong E_8$). Since S^* is nonmetacyclic, there is only one maximal subgroup of G containing S^* and so G/S^* must be cyclic. In particular, $G' \leq S^*$. Hence S^* is equal to one of the three maximal subgroups of S containing G'. They are E, E_1, and $S_0 = G'\langle a^{2^{m-1}}\rangle$. But S_0 is metacyclic (as a subgroup of F) and so $S_0/\Phi(G')$ cannot be elementary abelian of order 8. It follows that S^* is equal to E or E_1. Interchanging E and E_1, if necessary, we may assume that $S^* = E$ and so $E/\Phi(G') \cong E_8$. Since E is not metacyclic and G' is a metacyclic maximal subgroup of E, there is (by Proposition 2.1) an element x of order ≤ 4 in $E - G'$, as before, and we have here (in our case where $G/\Phi(G')$ has a normal E_8) in addition that $x^2 \in \Phi(G')$.

We have $W \leq G'$ and if G' is not a four-group, then also $W \leq R$. If $W \not\leq Z(G)$, then we always set $\langle z\rangle = W \cap Z(G)$. Since $x^2 \in G'$ and $o(x) \leq 4$, we have $x^2 \in W$ and, therefore, $[a, x^2] = z^\epsilon$, where $\epsilon = 0, 1$ and $\epsilon = 1$ if and only if a does not centralize x^2 (in which case $W \not\leq Z(G)$). We compute

$$z^\epsilon = [a, x^2] = [a, x][a, x]^x = vv^x$$

and so $v^x = v^{-1}z^\epsilon$.

We know that $a^{2^m} \in R - \Phi(G')$, $(ax)^{2^m} \in R - \Phi(G')$ and so $\langle a^{2^m}\rangle$ and $(ax)^{2^m}$ cover $G'/\langle v\rangle$ and, therefore, $G' = \langle a^{2^m}\rangle\langle v\rangle = \langle (ax)^{2^m}\rangle\langle v\rangle$. But $\langle a\rangle$ covers F/G' and $\langle ax\rangle$ covers F_1/G' and so $F = \langle a\rangle\langle v\rangle$ and $F_1 = \langle ax\rangle\langle v\rangle$. Set $b = [v, a]$ and $b_1 = [v, ax]$ so that $F' = \langle b\rangle$ and $F_1' = \langle b_1\rangle$, where we have used the facts that F and F_1 are metacyclic and $F = \langle a, v\rangle$, $F_1 = \langle ax, v\rangle$. Since $\langle b\rangle\langle b_1\rangle = R$, we may assume (interchanging $F = G'\langle a\rangle$ with $F_1 = G'\langle ax\rangle = G'\langle xa\rangle$, if necessary) that $b \in R - \Phi(G')$. Indeed, we have $[xa, x] = [a, x] = v$. Then we compute

$$\begin{aligned} b_1 = [v, ax] &= [v, x][v, a]^x = v^{-1}(x^{-1}vx)b^x = v^{-1}(v^{-1}z^\epsilon)b^x \\ &= v^{-2}z^\epsilon b^x \in R - \Phi(G'), \end{aligned}$$

since $v^{-2} \in \Phi(G')$, $b^x \in R - \Phi(G')$, and $z^\epsilon \in \Phi(G')$. Indeed, if $\Phi(G') \neq \{1\}$ and $W \not\leq Z(G)$, then $\langle z\rangle = W \cap Z(G) \leq \Phi(G')$. If $\Phi(G') = \{1\}$, then $|R| = 2$ and $R \leq Z(G)$ and so the fact that $x^2 \in R$ gives $\epsilon = 0$. Hence, in any case we get $b \in R - \Phi(G')$ and $b_1 \in R - \Phi(G')$ and (interchanging F and F_1, if necessary) we may assume that $o(b) = \exp(R)$.

Conjugating the relation $[v,a]=b$ (which gives $v^a=vb$ and $(v^{-2})^a=v^{-2}b^{-2}$) with x we get

$$\begin{aligned} b^x &= [v^{-1}z^\epsilon, a^x] = [v^{-1}, a(a^{-1}x^{-1}ax)] = [v^{-1}, av] = [v^{-1}, v][v^{-1}, a]^v \\ &= [v^{-1}, a] = v(a^{-1}v^{-1}a) = v(v^a)^{-1} = v(vb)^{-1} = b^{-1}, \end{aligned}$$

and so we get $b^x=b^{-1}$. From the above we also get

$$b_1 = v^{-2}z^\epsilon b^{-1} \quad \text{and so} \quad b_1^x = v^2 z^\epsilon b = b_1^{-1}.$$

Thus, x acts invertingly on R. We compute

$$b_1^a = (v^{-2})^a z^\epsilon (b^a)^{-1} = v^{-2}b^{-2}z^\epsilon (b^a)^{-1} = b^{-1}(v^{-2}z^\epsilon b^{-1})(b^a)^{-1} = b^{-1}b_1(b^a)^{-1}$$

and so $b_1^{-1}b_1^a = (bb^a)^{-1} \in \langle b_1\rangle \cap \langle b\rangle$, since $\langle b\rangle$ and $\langle b_1\rangle$ are normal subgroups of G.

We show that $\Phi(G)$ is a powerful 2-group and $(\Phi(G))' = \langle bb^a\rangle$. Indeed, we have $F=\langle a\rangle\langle v\rangle$, $a^2, v \in \Phi(G)$, $|F:\Phi(G)|=2$ and so $\Phi(G)=\langle a^2\rangle\langle v\rangle$. This gives $(\Phi(G))' = \langle [v,a^2]\rangle$ and since $[v,a^2]=[v,a][v,a]^a = bb^a$, we get $(\Phi(G))' = \langle bb^a\rangle$. But $\langle b\rangle = F'$ is normal in G and so $\langle b\rangle = \langle b^a\rangle$ and, therefore, $\langle bb^a\rangle \le \mho_1(\langle b\rangle)$. On the other hand, F is metacyclic and, therefore, b is a square in F and so $b=y^2$ for some $y\in F$. But F/G' is cyclic of order ≥ 4 and $b\in G'$ and so $y\in\Phi(G)$. It follows that $bb^a \in \mho_2(\langle y\rangle)$ and so $\Phi(G)' \le \mho_2(\Phi(G))$ and this means that $\Phi(G)$ is powerful (see the last sentence in Introduction).

We compute:

$$[a,x^2] = [a,x][a,x]^x = vv^x = v(v^{-1}z^\epsilon) = z^\epsilon$$

and so

$$(ax)^2 = axax = ax(xa)[a,x] = ax^2av = a^2x^2z^\epsilon v = a^2v(x^2z^\epsilon).$$

It is easy to see that $x^2z^\epsilon \in \Phi(\Phi(G))$. Indeed, $x^2z^\epsilon\in W$ and the facts that $a^{2^m}\in R-\Phi(G')$ and $|R:\Phi(G')|=2$ give $R=\langle a^{2^m}, \Phi(G')\rangle \le \Phi(\Phi(G))$ because $a^{2^{m-1}}\in\Phi(G)$, $(m\ge 2)$. Hence, if $W\le R$, we are done. If $W\not\le R$, then R is cyclic and we know that in that case $G'\cong E_4$ and so $|R|=2$. But then $x^2\in R\le Z(G)$ and so $\epsilon=0$ and again $a^{2^m}\in R-\Phi(G') = R-\{1\}$, and therefore, $\langle a^{2^m}\rangle = R \le \Phi(\Phi(G))$. We receive again $x^2z^\epsilon = x^2 \in R \le \Phi(\Phi(G))$. We have proved that in any case $x^2z^\epsilon\in\Phi(\Phi(G))$.

Since $(ax)^2 = a^2v(x^2z^\epsilon)$ and $x^2z^\epsilon\in\Phi(\Phi(G))$, we get

$$\Phi(G) = \langle v, a^2\rangle = \langle a^2v, a^2\rangle = \langle a^2v(x^2z^\epsilon), a^2\rangle = \langle (ax)^2, a^2\rangle.$$

But $\Phi(G)$ is powerful and so Proposition 2.11 implies $\Phi(G) = \langle (ax)^2\rangle\langle a^2\rangle$. We conclude:

$$G = \langle ax\rangle F = \langle ax\rangle(\Phi(G)\langle a\rangle) = \langle ax\rangle(\langle (ax)^2\rangle\langle a^2\rangle)\langle a\rangle$$
$$= (\langle ax\rangle\langle (ax)^2\rangle)(\langle a^2\rangle\langle a\rangle) = \langle ax\rangle\langle a\rangle.$$

In the rest of this section we make case-to-case investigations depending on the structure of G' and $G/\Phi(G')$. We shall use freely the notation and the results stated in Theorem 4.3.

Theorem 4.4: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup M and $d(G) = 2$. Suppose that G' is cyclic of order 2^n, $n > 1$. Then $G = EZ$, where E is normal in G and*

$$E = \langle v, x \mid v^{2^n} = 1,\, n > 1,\, x^2 \in \langle z\rangle, z = v^{2^{n-1}}, v^x = v^{-1}\rangle$$

is dihedral or generalized quaternion, $Z = \langle a\rangle$, $Z \cap E \le \langle z\rangle = Z(E)$, $|Z/(Z\cap E)| = 2^m$, $m > 1$, $[a, x] = v$, $v^a = v^{-1+4i}$ (i integer), and $[v, a^{2^{m-1}}] = 1$. Here $G' = \langle v\rangle \cong C_{2^n}$, $n > 1$, $\Phi(G) = G'\langle a^2\rangle$, $M = E\langle a^2\rangle$, $G = \langle ax\rangle\langle a\rangle$ is a product of two cyclic subgroups and $C_G(E) \not\le E$.

Proof. By Theorem 4.2, G has no normal E_8 and so we may use Theorem 4.3 (a) to (f). Since $v = [a, x] \in G' - R$, we have $G' = \langle v\rangle$. Also, $R = F'F_1'$ implies that interchanging $F = G'\langle a\rangle$ and $F_1 = G'\langle ax\rangle = G'\langle xa\rangle$ (and noting that $[xa, x] = [a, x] = v$), we may assume that $F' = R = \langle b\rangle$, where $b = [v, a]$ which gives $v^a = vb$, if necessary. Set $W \cap G' = \langle z\rangle = \Omega_1(R) = \Omega_1(G') \le Z(G)$. Since $o(x) \le 4$, $x^2 \in \langle z\rangle$ and so $x^2 = z^\eta$, $\eta = 0, 1$. Therefore

$$1 = [a, x^2] = [a, x][a, x]^x = vv^x \quad \text{and so} \quad v^x = v^{-1}.$$

It follows that E is dihedral or generalized quaternion. Since

$$(v^{2^{n-2}})^a = (vb)^{2^{n-2}} = v^{2^{n-2}}b^{2^{n-2}} = v^{2^{n-2}}z = v^{-2^{n-2}},$$

where $o(v^{2^{n-2}}) = 4$, it follows that $\langle a\rangle \cap E \le \langle z\rangle = Z(E)$.

Since $\langle v^{-2}\rangle = \langle b\rangle$, we may set $b = v^{-2+4i}$ for some integer i. We have $v^a = v^{-1+4i}$ and $v^{ax} = v^{1-4i} = vv^{-4i}$ so that $F_1 = \langle v\rangle\langle ax\rangle$ is ordinary metacyclic (since F_1 centralizes $\langle v\rangle/\mho_2(\langle v\rangle) = \langle v\rangle/\langle v^4\rangle$). By Proposition 2.10, F_1 is powerful. Since

$$(ax)^2 = axax = axxa[a, x] = ax^2av = a^2vz^\eta,$$

where $z^\eta \in \Phi(F_1)$, we get $F_1 = \langle ax, v\rangle = \langle ax, a^2\rangle = \langle ax\rangle\langle a^2\rangle$, where we have used Proposition 2.11. But then

$$G = F_1\langle a\rangle = (\langle ax\rangle\langle a^2\rangle)\langle a\rangle = \langle ax\rangle\langle a\rangle$$

and so G is a product of two cyclic subgroups.

Consider $E_1 = G'\langle xa^{2^{m-1}}\rangle$, where $E_1 \cap F = G'$ and so $W \not\leq E_1$. Hence, E_1 is a normal subgroup of G which does not possess a G-invariant four-subgroup. By Proposition 2.19, E_1 is of maximal class (since E_1 cannot be cyclic). But then $(xa^{2^{m-1}})^2 \in \langle z\rangle$ and, therefore, $xa^{2^{m-1}}$ also inverts $G' = \langle v\rangle$. Indeed, setting $xa^{2^{m-1}} = x'$, we get $G = \langle a, x'\rangle$ and so $[a, x'] = v' \in G' - R$ and $G' = \langle v'\rangle$. This gives $1 = [a, (x')^2] = [a, x'][a, x']^{x'} = v'(v')^{x'}$ and $(v')^{x'} = (v')^{-1}$. It follows that $a^{2^{m-1}}$ centralizes $\langle v\rangle$. Since $a^{2^{m-1}}$ does not fuse x and vx (noting that a fuses x and vx), there is $g \in \langle v\rangle$ with $ga^{2^{m-1}}$ centralizing x and $ga^{2^{m-1}}$ centralizes E and so $C_G(E) \not\leq E$. ∎

From Theorems 4.1, 4.2, 4.3(g13), and 4.4, we get the following important result.

Corollary 4.5: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Then $G = AB$ is a product of some cyclic subgroups A and B.*

The next two results are devoted to the case, where $G/\Phi(G')$ has no normal elementary abelian subgroup of order 8.

Theorem 4.6: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup M and $d(G) = 2$. Assume that $G' \cong E_4$ and G has no normal elementary abelian subgroup of order 8. Then G is a unique group of order 2^5*

$$G = \langle a, x \mid a^8 = x^4 = 1,\ a^4 = x^2 = z,\ [a, x] = v,\ v^2 = 1,\ [v, a] = z\rangle,$$

where $Z(G) = \langle z\rangle \cong C_2$, $G' = \langle z, v\rangle \cong E_4$, $M = \langle v\rangle \times \langle a^2, x\rangle \cong C_2 \times Q_8$ (and in fact this group is isomorphic to the group of Proposition 2.16(f)).

Proof. We may use Theorem 4.3 (a) to (g) (except (g6)). Here $|R| = 2$ so that $R = \langle z\rangle \leq Z(G)$ and $G' = \langle z\rangle \times \langle v\rangle$ with $v = [a, x]$. Since $x^2 \in R$, we have $v^x = v^{-1} = v$ and so $E = G'\langle x\rangle$ is abelian. But (by our assumption) E is not elementary abelian and so $x^2 = z$. We know that $b = [v, a] \in R - \{1\}$ and so $[v, a] = z$, $W = G' \not\leq Z(G)$ and $C_G(G') = M = E\langle a^2\rangle$. Also, $a^{2^m} \in R - \{1\}$ and so $a^{2^m} = z$. Since $(E\langle a^{2^{m-1}}\rangle)/R \cong E_8$, we have $[a^{2^{m-1}}, x] \leq R$.

If $[a^{2^{m-1}}, x] = 1$, then $i = xa^{2^{m-1}}$ is an involution in $M - F$ and so $\langle i \rangle \times G'$ is a normal elementary abelian subgroup of order 8, a contradiction. Hence $[a^{2^{m-1}}, x] = z$ and so $E\langle a^{2^{m-1}} \rangle = \langle v \rangle \times \langle x, a^{2^{m-1}} \rangle \cong C_2 \times Q_8$. We compute

$$[a^2, x] = [a, x]^a [a, x] = v^a v = (vz)v = z, \;\; [a^4, x] = [a^2, x]^{a^2} [a^2, x] = zz = 1,$$

and so if $m > 2$, then $\langle a^4 \rangle \geq \langle a^{2^{m-1}} \rangle$ and in that case $[a^{2^{m-1}}, x] = 1$, a contradiction. Hence $m = 2$ and the structure of G is uniquely determined. ∎

Theorem 4.7: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that G' is noncyclic, $\Phi(G') \neq \{1\}$, and $G/\Phi(G')$ has no normal elementary abelian subgroup of order 8. Then G' has a cyclic subgroup of index 2, $G/G' \cong C_4 \times C_2$, $\Phi(G)$ is abelian, and we have one of the following possibilities (depending on whether $Z(G)$ is noncyclic or cyclic):*

(a)

$$G = \langle a, x \mid a^8 = x^4 = 1, \, x^2 = u, \, a^4 = uz^\eta, \, \eta = 0, 1, \, [a, x] = v, \, v^{2^n} = 1, \, n \geq 2,$$
$$[u, a] = 1, \, v^{2^{n-1}} = z, \, v^x = v^{-1}, \, [v, a] = uv^{-2}z^\xi, \, \xi = 0, 1\rangle,$$

where $|G| = 2^{n+4}$, $Z(G) = \langle u, z \rangle \cong E_4$ and $G' = \langle u, v \rangle \cong C_2 \times C_{2^n}$.

(b)

$$G = \langle a, x \mid a^{16} = x^4 = 1, \, x^2 = u, \, a^4 = uv^{2^{n-2}} z^\eta, \, \eta = 0, 1, \, [a, x] = v,$$
$$v^{2^n} = 1, \, n \geq 4, \, u^a = uz, \, v^{2^{n-1}} = z, \, v^x = v^{-1}z, \, [u, v] = 1,$$
$$[v, a] = uv^{-2+2^{n-2}} z^\xi, \, \xi = 0, 1\rangle,$$

where $|G| = 2^{n+4}$, $Z(G) = \langle z \rangle \cong C_2$ and $G' = \langle u, v \rangle \cong C_2 \times C_{2^n}$.

Proof. We may use Theorem 4.3 (a) to (g) (except (g6)). Indeed, since $|G'| > 4$, Theorem 4.2 implies that G has no normal E_8. Applying Theorem 4.6 on the factor-group $G/\Phi(G')$, we get at once $m = 2$, i.e., $F/G' \cong C_4$ and $x^2 \in R - \Phi(G')$. But $x^2 = u$ is an involution and if C is a maximal subgroup of G' not containing u, then $C \cap R = \Phi(G')$, $G' = \langle u \rangle \times C$ and therefore C is cyclic of order 2^n, $n \geq 2$ (since G' is of rank 2). Hence, G' has a cyclic subgroup of index 2 and since $[a, x] = v \in G' - R$, we have $G' = \langle u \rangle \times \langle v \rangle$, $o(v) = 2^n$, $n \geq 2$, and $\Phi(G') = \langle v^2 \rangle$ with $R = \langle u \rangle \times \langle v^2 \rangle$. Let $\langle z \rangle = \mho_{n-1}(G')$ so that $z = v^{2^{n-1}}$, $W = \langle u, z \rangle \cong E_4$ and $z \in Z(G)$. It follows that E centralizes W and so $C_G(W) \geq M = E\langle a^2 \rangle$.

We know that $a^4 \in R - \Phi(G')$ and the element $b = [v, a]$ is of order $\exp(R) = 2^{n-1}$ so that $b = uv^{2i}$ for an odd integer i. Suppose that $C_{\langle v^2 \rangle}(a) > \langle z \rangle$ which forces $n \geq 3$. From $v^a = vb$, we get in this case

$$(v^{2^{n-2}})^a = (vb)^{2^{n-2}} = v^{2^{n-2}} b^{2^{n-2}} = v^{2^{n-2}} z = v^{-2^{n-2}}$$

since $v^{2^{n-2}}$ is an element of order 4 in $\langle v^2 \rangle$ and $v^{2^{n-1}} = z$. This is a contradiction and so $C_{\langle v^2 \rangle}(a) = \langle z \rangle$. In particular, $a^8 \in \langle z \rangle$ and so we have either $a^4 = uz^\eta$ (first case) or $n \geq 3$ and $a^4 = uv^{2^{n-2}} z^\eta$ (second case), where $\eta = 0, 1$.

Since $F' = \langle b \rangle$ and F is metacyclic, there is $y \in F$ such that $y^2 = b$. But $b \in G' - \Phi(G')$ and so $y \in F - G'$. The fact that $F/G' \cong C_4$ implies that $y \in G'a^2 \leq M$. Since G' is abelian, $C_{G'}(y) = C_{G'}(a^2)$ and so a^2 centralizes b. But $b = uv^{2i}$ (i odd) and $y \in M$ and so a^2 centralizes u which gives that a^2 centralizes $\langle v^{2i} \rangle = \langle v^2 \rangle$. On the other hand, $\langle v^2 \rangle = \Phi(G')$ is normal in G and so in case $n > 2$, a induces an involutory automorphism on $\langle v^2 \rangle$. From $v^a = vb$, we get $(v^2)^a = (vb)^2 = v^2 b^2$ and so $[v^2, a] = b^2 = v^{4i}$ (i odd). Hence, in case $n > 2$, a induces on $\langle v^2 \rangle$ an involutory automorphism such that $(\langle v^2, a \rangle)' = \langle v^4 \rangle$ and so $(v^2)^a = v^{-2} z^\zeta$, $\zeta = 0, 1$, where $\zeta = 1$ is possible only if $n > 3$. Thus $b^2 = v^{-4} z^\zeta$ which gives $v^{4i} = v^{-4} v^{\zeta 2^{n-1}}$ and so $4i \equiv -4 + \zeta 2^{n-1} \pmod{2^n}$ and therefore $2i \equiv -2 + \zeta 2^{n-2} \pmod{2^{n-1}}$. We get $b = uv^{2i} = uv^{-2+\zeta 2^{n-2} + \xi 2^{n-1}}$ (ξ an integer) and so $b = uv^{-2} v^{\zeta 2^{n-2}} z^\xi$, $\zeta = 0, 1$, $\xi = 0, 1$, and $\zeta = 1$ is possible only if $n > 3$.

In the first case, where $a^4 = uz^\eta$, we have $C_G(W) \geq \langle M, a \rangle = G$ and so $W \leq Z(G)$ which implies $\epsilon = 0$ and $v^x = v^{-1}$. It is easy to see that in this case $Z(G) = W \cong E_4$ (since x acts invertingly on G', $Z(G) \leq \Phi(G) = G'\langle a^2 \rangle$ and $[a^2, x] = v^a v = (vb)v = v^2 b \neq 1$). Suppose that in this case $\zeta = 1$, i.e., $b = uv^{-2} v^{2^{n-2}} z^\xi$, $n \geq 4$. Then we get

$$[a^2, x] = [a, x]^a [a, x] = v^a v = v^2 b = uv^{2^{n-2}} z^\xi,$$

and noting that a^2 centralizes v^2 follows

$$1 = [uz^\eta, x] = [a^4, x] = [a^2, x]^{a^2} [a^2, x] = (uv^{2^{n-2}} z^\xi)^{a^2} uv^{2^{n-2}} z^\xi = v^{2^{n-1}} = z,$$

a contradiction. Hence in this case $\zeta = 0$.

Suppose that we are in the second case, where $n \geq 3$ and $a^4 = uv^{2^{n-2}} z^\eta$. In this case we show first that $u^a = uz$ and so $W \not\leq Z(G)$, $x^2 = u \in W - Z(G)$ and $\epsilon = 1$, $v^x = v^{-1} z$. Indeed, $u = a^4 v^{-2^{n-2}} z^\eta$ and so

$$u^a = a^4 (v^{-2^{n-2}})^a z^\eta = a^4 v^{2^{n-2}} z^\eta = (a^4 v^{-2^{n-2}} z^\eta) z = uz.$$

Also, it is easy to show that in this case $n \geq 4$. If $n = 3$, then

$$b_1 = v^{-2}zb^{-1} = v^{-2}z(uv^{-2i}) = u(zv^{-2(1+i)}) \in W - \langle z \rangle$$

since $1 + i$ is even and so $v^{-2(1+i)} \in \langle z \rangle$. But then $\langle b_1 \rangle = F_1' \leq Z(G)$ and $W \leq Z(G)$, a contradiction. Assume that $\zeta = 0$ so that $b = uv^{-2}z^{\xi}$. But then $b_1 = v^{-2}zb^{-1} = uz^{\xi+1}$ is an involution in $W - \langle z \rangle$ and $\langle b_1 \rangle = F_1' \leq Z(G)$ which gives $W \leq Z(G)$, a contradiction. Hence, in this case we must have $\zeta = 1$. It is easy to see that in this case $Z(G) = \langle z \rangle$.

In both cases, using the obtained relations, we compute $bb^a = 1$ and so $(\Phi(G))' = \langle bb^a \rangle = \{1\}$ and $\Phi(G)$ is abelian. ∎

In what follows we may assume that G' is noncyclic of order > 4 and $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8.

Theorem 4.8: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that $G' \cong C_2 \times C_{2^n}$, $n \geq 2$, and $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8. Then we have:*

$$\begin{aligned} G = \langle a, x \mid [a, x] = v,\ & v^{2^n} = 1,\ n \geq 2,\ v^{2^{n-1}} = z,\ x^2 \in \langle z \rangle, \\ & [v, a] = uv^{2+4s}\ (s \text{ integer}),\ u^2 = [v, u] = 1,\ u^x = u,\ v^x = v^{-1}, \\ & a^{2^m} = uz^{\eta} \text{ or } a^{2^m} = uv^{2^{n-2}}z^{\eta},\ (\eta = 0, 1), \\ & \text{where } m \geq 2 \text{ and in the second case } n \geq 4,\ 1 + s \not\equiv 0 (\bmod 2^{n-3}), \\ & \text{and } n \geq m + 2 \rangle. \end{aligned}$$

Here $|G| = 2^{n+m+2}$, $n \geq 2$, $m \geq 2$, $G' = \langle u \rangle \times \langle v \rangle \cong C_2 \times C_{2^n}$, where in case $o(a) = 2^{m+1}$ we have $\langle u, z \rangle \leq Z(G)$ and so $Z(G)$ is noncyclic and in case $o(a) = 2^{m+2}$ we have $\langle u, z \rangle \not\leq Z(G)$ and so $Z(G)$ is cyclic in which case $n \geq 4$.

Proof. Since $|G'| \geq 8$, Theorem 4.2 implies that G has no normal E_8 and so we may use Theorem 4.3 (a) to (g). Since $[a, x] = v$ is of order 2^n, we get $G' = \langle u \rangle \times \langle v \rangle$ for some involution u, $\Phi(G') = \langle v^2 \rangle$, and $R = \langle u \rangle \times \langle v^2 \rangle$. By Theorem 4.3 (g6), $E/\Phi(G') \cong E_8$ and $x^2 \in \langle z \rangle$, where we set $v^{2^{n-1}} = z$ and so $\langle z \rangle = \Omega_1(\Phi(G')) \leq Z(G)$. This gives $\epsilon = 0$ and $v^x = v^{-1}$ and therefore x acts invertingly on G'. We have $W = \langle u \rangle \times \langle z \rangle$ and since $|G : C_G(W)| \leq 2$ and $W = Z(E)$, we have $C_G(W) \geq M = E\langle a^2 \rangle$. Also, $F/G' \cong C_{2^m}$, $m \geq 2$.

Since $b = [v, a] \in R - \Phi(G')$ is of order $\exp(R) = 2^{n-1}$, we may set $b = uv^{2i}$ with an odd integer i and we may also write $b = uv^{2+4s}$ (s integer). Suppose that $C_{\langle v^2\rangle}(a) > \langle z\rangle$ which implies $n \geq 3$. From $b = [v, a]$ we get $v^a = vb$ and so

$$(v^{2^{n-2}})^a = (vb)^{2^{n-2}} = v^{2^{n-2}}(uv^{2i})^{2^{n-2}} = v^{2^{n-2}}z = v^{-2^{n-2}},$$

where $o(v^{2^{n-2}}) = 4$. This is a contradiction and so $C_{\langle v^2\rangle}(a) = \langle z\rangle$. But we know that $a^{2^m} \in R - \Phi(G')$ and so $a^{2^{m+1}} \in \langle z\rangle$ which gives either $a^{2^m} = uz^\eta$ or $n \geq 3$ and $a^{2^m} = uv^{2^{n-2}}z^\eta$, where $\eta = 0, 1$. If $a^{2^m} = uz^\eta$, then $o(a) = 2^{m+1}$, $C_G(W) \geq \langle M, a\rangle = G$ and so $W \leq Z(G)$ and $Z(G)$ is noncyclic.

Suppose that we are in the second case, where $n \geq 3$, $a^{2^m} = uv^{2^{n-2}}z^\eta$, $\eta = 0, 1$, and $o(a) = 2^{m+2}$. In this case $u = a^{2^m}v^{-2^{n-2}}z^\eta$ and so

$$u^a = a^{2^m}v^{2^{n-2}}z^\eta = (a^{2^m}v^{-2^{n-2}}z^\eta)z = uz,$$

which implies that $W \not\leq Z(G)$ and so $Z(G)$ is cyclic. In this case we must have $n \geq 4$. Indeed, if $n = 3$, then

$$b_1 = v^{-2}b^{-1} = v^{-2}(uv^{-2i}) = uv^{-2(1+i)}$$

and so the fact that $1 + i$ is even and $o(v) = 8$ implies $v^{-2(1+i)} \in \langle z\rangle$. Hence b_1 is an involution in $W - \langle z\rangle$ and since $\langle b_1\rangle = F_1' \leq Z(G)$, we get $W \leq Z(G)$, a contradiction. Since

$$b_1 = v^{-2}b^{-1} = v^{-2}uv^{-2-4s} = uv^{-4(1+s)}$$

and $b_1 \in R - \Phi(G')$ cannot be an involution ($Z(G)$ is cyclic), it follows $1 + s \not\equiv 0 (\mathrm{mod}\ 2^{n-3})$. We have $G = \langle a\rangle\langle ax\rangle$, where $a^{2^m} = uv^{2^{n-2}}z^\eta$ and $a^{2^{m+1}} = z$. Since $o(a^{2^m}) = 4$ and a^{2^m} is inverted by x, it follows that

$$\langle a^{2^m}\rangle \not\leq Z(G) \quad \text{and so} \quad \langle a^{2^m}\rangle \not\leq \langle a\rangle \cap \langle ax\rangle \leq Z(G).$$

Since $(ax)^{2^m} \in R - \Phi(G')$ cannot be an involution (because $W \not\leq Z(G)$), it follows that $\langle (ax)^{2^m}\rangle$ being distinct from $\langle a^{2^m}\rangle$, $o((ax)^{2^m}) \geq 8$ and so $\langle (ax)^{2^{m+1}}\rangle \leq \langle v^2\rangle$ and $\langle (ax)^{2^{m+1}}\rangle > \langle z\rangle$. This implies that $\langle a\rangle \cap \langle ax\rangle = \langle z\rangle$ and so $o(a) = 2^{m+2}$ and $|G| = 2^{n+m+2}$ gives $o(ax) = 2^{m+1+r}$, where $o((ax)^{2^{m+1}}) = 2^r$, $r \geq 2$. This implies (by the product formula) $m + r = n$ and so $n \geq m + 2$. ∎

In the rest of this section we consider the remaining case, where G' has no cyclic subgroup of index 2. By Theorems 4.2 and 4.7, G has no normal elementary abelian subgroup of order 8 but $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8. We shall use freely the notation and all results from Theorem 4.3.

THEOREM 4.9: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that $G' \cong C_{2^r} \times C_{2^r}$, $r \geq 2$, is homocyclic. Then $G/G' \cong C_2 \times C_4$ and*

$$G = \langle a, x \mid a^{2^{r+2}} = 1,\ r \geq 2,\ [a, x] = v,\ [v, a] = b,\ v^{2^r} = b^{2^r} = [v, b] = 1,\\ v^{2^{r-1}} = u,\ b^{2^{r-1}} = z,\ x^2 \in \langle u, z\rangle,\ b^x = b^{-1},\ v^x = v^{-1}z^{\epsilon},\ \epsilon = 0, 1,\\ \text{and } \epsilon = 1 \text{ if and only if } x^2 \notin \langle z\rangle,\ b^a = b^{-1}z^{\eta},\ \eta = 0, 1,\\ a^4 = v^{-2}b^{-1}u^{\eta}z^{\zeta},\ \zeta = 0, 1\rangle.$$

Here $|G| = 2^{2r+3}$, $r \geq 2$, $G' = \langle v\rangle \times \langle b\rangle \cong C_{2^r} \times C_{2^r}$, $Z(G) = \langle z\rangle \cong C_2$, and $(\Phi(G))' = \langle z^{\eta}\rangle$, where $\Phi(G) = \langle a^2\rangle\langle v\rangle$.

Proof. The element $v = [a, x] \in G' - R$ is of order 2^r and if $\langle b'\rangle \cong C_{2^r}$ is a complement of $\langle v^2\rangle$ in R, then $R = \langle b'\rangle \times \langle v^2\rangle$, $\Phi(G') = \langle (b')^2\rangle \times \langle v^2\rangle$ and all elements in $R - \Phi(G')$ are of order 2^r. Hence, $b = [v, a] \in R - \Phi(G')$ is of order 2^r and so $G' = \langle b\rangle \times \langle v\rangle$, $R = \langle b\rangle \times \langle v^2\rangle$, $\Phi(G') = \langle b^2\rangle \times \langle v^2\rangle$. We set $v^{2^{r-1}} = u$, $b^{2^{r-1}} = z$ so that $W = \langle u, z\rangle$, $\langle z\rangle = \mho_{r-1}(R) \leq Z(G)$ and we know that x acts invertingly on R. From $b = [v, a]$ follows $v^a = vb$, $(v^{2^{r-1}})^a = v^{2^{r-1}}b^{2^{r-1}}$ and $u^a = uz$ so that $W \not\leq Z(G)$ and $W \cap Z(G) = \langle z\rangle$. Since $W \leq Z(E)$, we get $C_G(W) = M = E\langle a^2\rangle$. We have $x^2 \in W$ and $v^x = v^{-1}z^{\epsilon}$, $\epsilon = 0, 1$, where $\epsilon = 1$ if and only if $x^2 \notin \langle z\rangle$. It follows $Z(E) = W$.

We have $b_1 = v^{-2}z^{\epsilon}b^{-1} \in R - \Phi(G')$, where $o(b_1) = 2^r$, $\langle b_1\rangle = F_1'$, $F_1 = G'\langle ax\rangle$, and $b_1^{2^{r-1}} = (b^{-1})^{2^{r-1}} = z$. But $R = \langle b\rangle\langle b_1\rangle$, $|R| = 2^{2r-1}$, and so (by the product formula) $\langle b\rangle \cap \langle b_1\rangle = \langle z\rangle$. Since $bb^a \in \langle b\rangle \cap \langle b_1\rangle$ (Theorem 4.3 (g11)), we get $b^a = b^{-1}z^{\eta}$, $\eta = 0, 1$, and $(\Phi(G))' = \langle bb^a\rangle = \langle z^{\eta}\rangle$. Hence, $\Phi(G) = G'\langle a^2\rangle = \langle a^2\rangle\langle v\rangle$ is either abelian or minimal nonabelian (Proposition 2.9). Also, $b_1^{-1}(b_1)^a = (bb^a)^{-1} = z^{\eta}$ and $(b_1)^a = b_1z^{\eta}$ which gives

$$(b_1u^{\eta})^a = b_1z^{\eta}(uz)^{\eta} = b_1u^{\eta} = v^{-2}b^{-1}u^{\eta}z^{\epsilon}$$

so that $C_{G'}(a) = \langle v^{-2}b^{-1}u^{\eta}z^{\epsilon}\rangle$ is of order 2^r (noting that $G' = \langle b_1u^{\eta}\rangle \times \langle v\rangle$ and $u^a = uz$ forces $C_{\langle v\rangle}(a) = \{1\}$). But $a^{2^m} \in R - \Phi(G')$ is of order 2^r and so $\langle a^{2^m}\rangle = \langle v^{-2}b^{-1}u^{\eta}z^{\epsilon}\rangle$ which gives $\langle a^{2^{m+1}}\rangle = \langle v^{-4}b^{-2}\rangle = \langle v^4b^2\rangle$. We claim that for all $s \geq 1$, $[a^{2^s}, x] = v^{2^s}b^{2^{s-1}}$. Indeed,

$$[a^2, x] = [a, x]^a[a, x] = v^av = (vb)v = v^2b$$

and using the facts that $v^2 \in \Phi(\Phi(G)) \leq Z(\Phi(G))$ and $b^{a^2} = (b^{-1}z^\eta)^a = b$, we get

$$[a^4, x] = [a^2, x]^{a^2}[a^2, x] = (v^2b)^{a^2}v^2b = v^4b^2.$$

Assuming $s > 2$ and using the induction on s (since $a^{2^{s-1}} \in Z(\Phi(G))$), we get

$$[a^{2^s}, x] = [a^{2^{s-1}}a^{2^{s-1}}, x] = [a^{2^{s-1}}, x]^{a^{2^{s-1}}}[a^{2^{s-1}}, x]$$
$$= (v^{2^{s-1}}b^{2^{s-2}})^{a^{2^{s-1}}}(v^{2^{s-1}}b^{2^{s-2}}) = (v^{2^{s-1}}b^{2^{s-2}})^2 = v^{2^s}b^{2^{s-1}}.$$

Since $a^{2^m} \in R - \Phi(G')$, $m \geq 2$, and x acts invertingly on R, we get

$$(a^{2^m})^x = a^{-2^m} \quad \text{and so} \quad [a^{2^m}, x] = a^{-2^{m+1}}.$$

By the above relation, $[a^{2^m}, x] = v^{2^m}b^{2^{m-1}} = a^{-2^{m+1}}$ and so using a result from the previous paragraph we get $\langle v^{2^m}b^{2^{m-1}}\rangle = \langle v^4b^2\rangle$ which forces $m = 2$. We have proved that G/G' is abelian of type $(4, 2)$ and so $a^4 \in R - \Phi(G')$.

Because $\langle a^4\rangle = \langle v^{-2}b^{-1}u^\eta z^\epsilon\rangle$, we get

$$a^4 = v^{-2}b^{-1}u^\eta z^\epsilon(v^{-2}b^{-1}u^\eta z^\epsilon)^{2i}$$

for some integer i, and so $a^4 = v^{-2-4i}b^{-1-2i}u^\eta z^\epsilon$. Then (noting that $a^8 = v^{-4}b^{-2}$) $a^8 = v^{-4-8i}b^{-2-4i} = v^{-4}b^{-2}$ implies $i \equiv 0 \pmod{2^{r-2}}$ and so $a^4 = v^{-2}b^{-1}z^{-i}u^\eta z^\epsilon$ and $a^4 = v^{-2}b^{-1}u^\eta z^\zeta$, $\zeta = 0, 1$. We see also $Z(G) = \langle z\rangle$. ∎

THEOREM 4.10: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that G' has no cyclic subgroup of index 2. Then $Z(G)$ is elementary abelian of order at most 4, $Z(G) \leq \Omega_1(G')$, and $G/G' \cong C_4 \times C_2$.*

Proof. We consider $G/\mho_2(G')$, where $G'/\mho_2(G') = (G/\mho_2(G'))' \cong C_4 \times C_4$ and so by Theorem 4.9, $G/G' \cong C_4 \times C_2$. We may use Theorem 4.3 with $m = 2$. We have $Z(G) \leq \Phi(G) = G'\langle a^2\rangle$, where $|(G'\langle a^2\rangle) : G'| = 2$ and $W \leq \Phi(G')$. Note that x acts invertingly on R. If x commutes with an element $y \in G' - R$, then $y^2 \in R$ must be an involution and so $\exp(G') = 4$ and $G' \cong C_4 \times C_4$. But in that case (Theorem 4.9), $Z(G) \cong C_2$ and $Z(G) \leq W$ and we are done. Hence, we may assume that $C_{G'}(x) = C_R(x) = W$ and so $Z(G) \cap G' \leq W$. Suppose that there is an element $l \in \Phi(G) - G'$ such that $l \in Z(G)$. We have $l^2 \in G'$ and so $l^2 \in R$ and therefore l^2 must be an involution in W (since $\Omega_1(\Phi(G)) = W$). But $W \leq \Phi(G')$ and so there is an element $k \in G'$ such that $k^2 = l^2$. In that case, kl is an involution in $\Phi(G) - G'$, a contradiction. Hence, $Z(G) \leq W$ and we are done. ∎

Theorem 4.11: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that $G' \cong C_{2^r} \times C_{2^{r+1}}$, $r \geq 2$. Then we have*

$$G = \langle a, x \mid a^{2^{r+2}} = 1,\, r \geq 2,\, [a,x] = v,\, [v,a] = b,\, v^{2^{r+1}} = b^{2^r} = [v,b] = 1, \\ v^{2^r} = z,\, b^{2^{r-1}} = u,\, x^2 \in \langle u, z\rangle \cong E_4,\, b^x = b^{-1},\, v^x = v^{-1}, \\ b^a = b^{-1},\, a^4 = v^{-2}b^{-1}w,\, w \in \langle u, z\rangle\rangle.$$

Here $|G| = 2^{2r+4}$, $r \geq 2$, $G' = \langle b\rangle \times \langle v\rangle \cong C_{2^r} \times C_{2^{r+1}}$, $Z(G) = \langle u, z\rangle \cong E_4$, and $\Phi(G) = G'\langle a^2\rangle$ is abelian.

Proof. By Theorem 4.10, we have $m = 2$ in Theorem 4.3. The element $v = [a,x] \in G' - R$ is of order 2^{r+1} so that R is homocyclic of rank 2 and exponent 2^r. It follows

$$R = \langle b\rangle \times \langle v^2\rangle = \langle b_1\rangle \times \langle v^2\rangle = \langle b\rangle \times \langle b_1\rangle$$

and so $\epsilon = 0$, $W = \langle v^{2^r}\rangle \times \langle b^{2^{r-1}}\rangle = Z(G)$, x acts invertingly on G', $x^2 \in W$, $b^a = b^{-1}$ and $b_1^a = b_1$, where $v^a = vb$ and $b_1 = v^{-2}b^{-1}$. We set $v^{2^r} = z$ and $b^{2^{r-1}} = u$. Since $C_{\langle v\rangle}(a) = \langle z\rangle$, we have $C_{G'}(a) = \langle b_1\rangle \times \langle z\rangle$. On the other hand, $a^4 \in R - \Phi(G')$ is of order 2^r and so $\langle a^4\rangle$ is a cyclic subgroup of index 2 in $C_{G'}(a)$. This gives

$$\langle a^4\rangle = \langle b_1 z^\zeta\rangle = \langle v^{-2}b^{-1}z^\zeta\rangle,\ \zeta = 0, 1.$$

Also, $(\Phi(G))' = \langle bb^a\rangle = \{1\}$ and therefore $\Phi(G) = G'\langle a^2\rangle$ is abelian.

We get

$$a^4 = v^{-2}b^{-1}z^\zeta(v^{-2}b^{-1}z^\zeta)^{2i} = v^{-2-4i}b^{-1-2i}z^\zeta,$$

where i is an integer. On the other hand,

$$[a^2, x] = [a,x]^a[a,x] = v^a v = v^2 b \quad \text{and}$$

$$[a^4, x] = [a^2, x]^{a^2}[a^2, x] = (v^2b)^{a^2}(v^2b) = (v^2b)^2 = v^4b^2.$$

Since x inverts G', $[a^4, x] = a^{-8}$ and so $a^8 = v^{-4}b^{-2}$. This gives

$$a^8 = v^{-4}b^{-2} = v^{-4-8i}b^{-2-4i} \quad \text{and} \quad -2i \equiv 0 \pmod{2^{r-1}},$$

which implies $a^4 = v^{-2}b^{-1}w$ with $w \in \langle u, z\rangle$ since $v^{-4i} \in \langle z\rangle$ and $b^{-2i} \in \langle u\rangle$. ∎

Somewhat more difficult is the next special case, where $G' \cong C_{2^r} \times C_{2^{r+2}}$, $r \geq 2$. After that we shall be able to investigate the general case.

THEOREM 4.12: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that $G' \cong C_{2^r} \times C_{2^{r+2}}$, $r \geq 2$. Then we have*

$$G = \langle a, x \mid a^{2^{r+2}} = 1,\ r \geq 2,\ [a,x] = v,\ [v,a] = b,\ v^{2^{r+2}} = b^{2^{r+1}} = [v,b] = 1,$$
$$v^{2^{r+1}} = b^{2^r} = z,\ v^{2^r} b^{2^{r-1}} = u,\ x^2 \in \langle u, z\rangle \cong E_4,$$
$$b^x = b^{-1},\ v^x = v^{-1},\ b^a = b^{-1},\ a^4 = v^{-2}b^{-1}w,\ w \in \langle u, z\rangle\rangle.$$

Here $|G| = 2^{2r+5}$, $r \geq 2$, $G' = \langle b, v\rangle \cong C_{2^r} \times C_{2^{r+2}}$, $Z(G) = \langle u, z\rangle \cong E_4$, and $\Phi(G) = G'\langle a^2\rangle$ is abelian.

Proof. We use freely Theorem 4.10 and 4.3 with $m = 2$. The element $v = [a,x] \in G' - R$ is of order 2^{r+2} and we set $z = v^{2^{r+1}}$ so that $z \in Z(G)$ since $\langle z\rangle = \mho_{r+1}(G')$. The element $b = [v,a] \in R - \Phi(G')$ is of order $\exp(R) = 2^{r+1}$ and since $\langle b\rangle$ covers $R/\langle v^2\rangle \cong C_{2^r}$, we get $\langle b\rangle \cap \langle v^2\rangle = \langle b\rangle \cap \langle v\rangle = \langle z\rangle$. We have $b_1 = [v, ax] \in R - \Phi(G')$ and $\langle b_1\rangle$ also covers $R/\langle v^2\rangle$. We know that $b_1 = v^{-2}z^{\epsilon}b^{-1}$ and so $b_1^{2^r} = v^{-2^{r+1}}b^{-2^r} = zz = 1$ and therefore b_1 is of order 2^r. Thus $\langle b_1\rangle \cap \langle v^2\rangle = \{1\}$ and so $R = \langle b_1\rangle \times \langle v^2\rangle = \langle b_1\rangle \times \langle b\rangle$. Set $u = b_1^{2^{r-1}} = v^{-2^r}b^{-2^{r-1}} = v^{2^r}b^{2^{r-1}}$ so that $W = \langle u, z\rangle \cong Z(G)$ (noting that $\langle b_1\rangle = F_1'$ is normal in G and so $u \in Z(G)$), which gives $\epsilon = 0$ and $v^x = v^{-1}$. We have $x^2 \in W$ and we know that x acts invertingly on R and so $b^x = b^{-1}$. We have also $b_1^{-1}b_1^a = (bb^a)^{-1} \in \langle b\rangle \cap \langle b_1\rangle = \{1\}$ and so $b^a = b^{-1}$, $b_1^a = b_1$, and $(\Phi(G))' = \langle bb^a\rangle = \{1\}$. Since $v^a = vb$, we get $(v^{2^r})^a = v^{2^r}b^{2^r} = v^{2^r}z = v^{-2^r}$ (since $(v^{2^r})^2 = z$) and so $C_{G'}(a) = \langle b_1\rangle \times \langle z\rangle$. On the other hand, $a^4 \in R - \Phi(G')$, $\langle a^4\rangle$ covers $R/\langle v^2\rangle$ and $a^4 \in C_{G'}(a)$ which gives $\langle a^4\rangle = \langle b_1 z^{\eta}\rangle = \langle v^{-2}b^{-1}z^{\eta}\rangle$, $\eta = 0, 1$.

We compute (noting that x inverts a^4):

$$[a^2, x] = [a,x]^a[a,x] = v^a v = v^2 b,\ [a^4, x] = a^{-8} = [a^2,x]^{a^2}[a^2,x] = (v^2b)^{a^2}(v^2b)$$
$$= v^4b^2$$

and so $a^8 = v^{-4}b^{-2}$. Using the last result from the previous paragraph, we get

$$a^4 = v^{-2}b^{-1}z^{\eta}(v^{-2}b^{-1}z^{\eta})^{2i} = v^{-2-4i}b^{-1-2i}z^{\eta} \quad (i \text{ integer}),$$

$a^8 = v^{-4}b^{-2} = v^{-4-8i}b^{-2-4i}$ and so $v^{-8i}b^{-4i} = 1$. This gives $i \equiv 0 \pmod{2^{r-2}}$ since $\langle v\rangle \cap \langle b\rangle = \langle z\rangle$ and $v^{2^{r+1}}b^{2^r} = z^2 = 1$. We may set $i = \zeta 2^{r-2}$ (ζ integer) and then

$$a^4 = v^{-2}b^{-1}z^{\eta}(v^{-2^r}b^{-2^{r-1}})^{\zeta} = v^{-2}b^{-1}z^{\eta}u^{\zeta}, \quad \zeta = 0, 1. \qquad ∎$$

Finally, we consider the general case, where $G' \cong C_{2^r} \times C_{2^{r+s+1}}$ with $r \geq 2$ and $s \geq 2$.

THEOREM 4.13: *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup and $d(G) = 2$. Assume that $G' \cong C_{2^r} \times C_{2^{r+s+1}}$, $r \geq 2$, $s \geq 2$. Then we have one of the following two possibilities (depending on $Z(G)$ being noncyclic or cyclic):*

(a)

$$G = \langle a, x \,|\, a^{2^{r+2}} = 1,\, r \geq 2,\, [a, x] = v,\, [v, a] = b,\, v^{2^{r+s+1}} = b^{2^{r+s}} = [v, b] = 1,$$
$$s \geq 2,\, b^{2^r} = v^{-2^{r+1}},\, v^{2^{r+s}} = z,\, a^{2^{r+1}} = u,\, x^2 \in W = \langle u, z\rangle \cong E_4,$$
$$b^a = b^{-1},\, b^x = b^{-1},\, v^x = v^{-1},\, a^4 = v^{-2}b^{-1}w,\, w \in W\rangle.$$

Here $|G| = 2^{2r+s+4}$, $r \geq 2$, $s \geq 2$, $G' = \langle b, v\rangle \cong C_{2^r} \times C_{2^{r+s+1}}$, $\langle b\rangle \cap \langle v\rangle \cong C_{2^s}$, $Z(G) = W = \langle u, z\rangle \cong E_4$, and $\Phi(G) = G'\langle a^2\rangle$ is abelian.

(b)

$$G = \langle a, x \,|\, a^{2^{r+3}} = 1,\, r \geq 2,\, [a, x] = v,\, [v, a] = b,\, v^{2^{r+s+1}} = b^{2^{r+s}} = [v, b] = 1,$$
$$s \geq 2,\, v^{2^{r+s}} = a^{2^{r+2}} = z,\, b^{2^r} = v^{-2^{r+1}}z,\, u = v^{-2^r(1+2^{s-1})}b^{-2^{r-1}},$$
$$u^a = uz,\, x^2 \in W = \langle u, z\rangle \cong E_4,\, b^a = b^{-1}z^{\delta},\, \delta = 0, 1,$$
$$b^x = b^{-1},\, v^x = v^{-1}z^{\epsilon},\, \epsilon = 0, 1,\, \epsilon = 0 \text{ if and only if} x^2 \in \langle z\rangle,$$
$$a^4 = v^{-2}b^{-1}u^{\delta}z^{\tau},\, \tau = 0, 1\rangle.$$

Here $|G| = 2^{2r+s+4}$, $r \geq 2$, $s \geq 2$, $G' = \langle b, v\rangle \cong C_{2^r} \times C_{2^{r+s+1}}$, $\langle b\rangle \cap \langle v\rangle \cong C_{2^s}$, $Z(G) = \langle z\rangle \cong C_2$, $\Phi(G) = G'\langle a^2\rangle$, $(\Phi(G))' = \langle z^{\delta}\rangle$ and so $\Phi(G)$ is either abelian ($\delta = 0$) or minimal nonabelian ($\delta = 1$).

Proof. We use freely Theorem 4.3 with $m = 2$ and $Z(G) \leq W$ (see Theorem 4.10). The element $v = [a, x] \in G' - R$ is of order 2^{r+s+1} and the element $b = [v, a] \in R - \Phi(G')$ is of order $\exp(R) = 2^{r+s}$. Since $\langle b\rangle$ covers $R/\langle v^2\rangle \cong C_{2^r}$, we have $c = b^{2^r} \in \langle v^2\rangle$, $o(c) = 2^s$ and $\langle v^2\rangle/\langle c\rangle \cong C_{2^r}$. Set $z = v^{2^{r+s}}$ so that $\langle z\rangle = \mho_{r+s}(G') \leq Z(G)$ and $\langle z\rangle < \langle c\rangle$. We know that the element $x \in E - G'$ (with $x^2 \in W = \Omega_1(G')$) inverts each element in R and $v^x = v^{-1}z^{\epsilon}$, where $\epsilon = 0, 1$ and $\epsilon = 1$ if and only if $x^2 \in W - Z(G)$. We note that $W \leq Z(E)$

and so $C_G(W) \geq M = E\langle a^2 \rangle$. Also, $b_1 = [v, ax] = v^{-2}b^{-1}z^{\epsilon}$, $b^x = b^{-1}$ and $b_1^x = b_1^{-1}$.

Set $S = \langle c \rangle$ and $S^* = \mho_r(G') = \langle v^{2^r} \rangle$ so that S^* is normal in G and $|S^* : S| = 2$. We have

$$(v^{2^{r+s-1}})^a = v^{2^{r+s-1}} b^{2^{r+s-1}} = v^{2^{r+s-1}} z = v^{-2^{r+s-1}}$$

since $v^{2^{r+s-1}}$ is an element of order 4. This gives $C_{\langle v \rangle}(a) = \langle z \rangle$. Also, $(v^{2^r})^a = (bv)^{2^r} = cv^{2^r}$. Since $a^4 \in R - \Phi(G')$ and $\langle a^4 \rangle$ covers $R/\langle v^2 \rangle$, we have either $o(a) = 2^{r+2}$ with $\langle a \rangle \cap \langle v \rangle = \{1\}$ or $o(a) = 2^{r+3}$ with $\langle a \rangle \cap \langle v \rangle = \langle z \rangle$. On the other hand, $(ax)^4 \in R - \Phi(G')$ and $\langle (ax)^4 \rangle$ covers $R/\langle v^2 \rangle$ so that $o(ax) = 2^{r+2+t}$, where $2^t = |\langle ax \rangle \cap \langle v^2 \rangle|$. Since $|G| = 2^{2r+s+4}$, $G = \langle a \rangle \langle ax \rangle$ and $o(a) \leq 2^{r+3}$, it follows that $o(ax) = 2^{r+2+t} \geq 2^{r+s+1}$ and so $t \geq s - 1$. By our assumption, $s \geq 2$ and so $t \geq 1$, which implies that $\langle ax \rangle \geq \langle z \rangle$. If $\langle a \rangle \cap \langle v \rangle = \{1\}$, then $\langle a \rangle \cap \langle ax \rangle = \{1\}$, $o(ax) = 2^{r+s+2}$ and therefore $t = s$. If $\langle a \rangle \cap \langle v \rangle = \langle z \rangle$, then $\langle a \rangle \cap \langle ax \rangle = \langle z \rangle$, $o(a) = 2^{r+3}$ and so again $t = s$. In any case, $\langle ax \rangle \cap \langle v^2 \rangle = S = \langle c \rangle$ and so ax centralizes c. But x inverts c and so $c^a = c^{-1}$. From $(v^{2^r})^a = cv^{2^r}$ follows

$$(v^{2^r})^{a^2} = c^a (v^{2^r})^a = c^{-1}(cv^{2^r}) = v^{2^r}.$$

Hence a induces an involutory automorphism on $S^* = \langle v^{2^r} \rangle$ with $c^a = c^{-1}$, where $o(c) \geq 4$ and $|S^* : \langle c \rangle| = 2$. This gives $(v^{2^r})^a = cv^{2^r} = v^{-2^r}z^{\zeta}$, $\zeta = 0, 1$, and so $c = b^{2^r} = v^{-2^{r+1}}z^{\zeta}$, where $o(v^{2^r}) = 2^{s+1} \geq 8$.

We compute $b_1^{2^r} = (v^{-2}b^{-1}z^{\epsilon})^{2^r} = v^{-2^{r+1}}v^{2^{r+1}}z^{\zeta} = z^{\zeta}$ and so $\langle b \rangle \cap \langle b_1 \rangle = \langle z^{\zeta} \rangle$. From Theorem 4.3(g11) follows

$$b_1^{-1}b_1^a = (bb^a)^{-1} \in \langle z^{\zeta} \rangle \quad \text{and so} \quad b^a = b^{-1}z^{\delta},\ b_1^a = b_1 z^{\delta},\ \delta = 0, 1,$$

where $\zeta = 0$ implies $\delta = 0$. Also, $(\Phi(G))' = \langle bb^a \rangle = \langle z^{\delta} \rangle$ and, therefore, $\Phi(G)$ is either abelian or minimal nonabelian.

We get $b^{a^2} = (b^{-1}z^{\delta})^a = b$ and $(v^2)^{a^2} = v^2$ since $v^2 \in \Phi(\Phi(G)) \leq Z(\Phi(G))$, where $\Phi(G) = G'\langle a^2 \rangle$. Also we know that x acts invertingly on R and $a^4 \in R - \Phi(G')$ and all this gives:

$$[a^2, x] = [a, x]^a [a, x] = v^a v = (vb)v = v^2 b,$$

$$[a^4, x] = a^{-8} = [a^2, x]^{a^2}[a^2, x] = (v^2 b)^{a^2}(v^2 b) = (v^2 b)^2 = v^4 b^2,$$

and so $a^8 = v^{-4}b^{-2}$. From this result also follows

$$(a^8)^{2^{r-1}} = a^{2^{r+2}} = v^{-2^{r+1}}b^{-2^r} = v^{-2^{r+1}}v^{2^{r+1}}z^{\zeta} = z^{\zeta}$$

and since $\langle ax \rangle \cap \langle v \rangle = \langle c \rangle \cong C_{2^s}$, we get $\langle ax \rangle \cap \langle a \rangle = \langle z^{\zeta} \rangle$.

Suppose $\zeta = 0$. In that case $b^{2^r} = v^{-2^{r+1}}$, $\langle b\rangle \cap \langle b_1\rangle = F' \cap F'_1 = \{1\}$, $\delta = 0$, $o(b_1) = 2^r$, $a^{2^{r+1}} = u \in Z(G)$ and so $W = \langle u, z\rangle = Z(G)$, $\epsilon = 0$, $v^x = v^{-1}$, $b^a = b^{-1}$, $b_1^a = b_1$, and $(\Phi(G))' = \{1\}$. Since $C_{G'}(a) = \langle b_1\rangle \times \langle z\rangle$ and $b_1 = v^{-2}b^{-1}$, we get $\langle a^4\rangle = \langle v^2bz^{\xi}\rangle$, $\xi = 0, 1$. Hence

$$a^4 = v^2bz^{\xi}(v^2bz^{\xi})^{2i} = v^{2+4i}b^{1+2i}z^{\xi} \quad (i \text{ integer})$$

and so using a result from the previous paragraph, we get

$$a^8 = v^{-4}b^{-2} = v^{4+8i}b^{2+4i},\ v^{8+8i}b^{4+4i} = 1,\ i + 1 \equiv 0 \ (\text{mod}\ \ 2^{r-2}),$$

and we set $i = -1 + t2^{r-2}$ (t an integer). This gives

$$a^4 = v^{-2+t2^r}b^{-1+t2^{r-1}}z^{\xi} = v^{-2}b^{-1}(v^{2^r}b^{2^{r-1}})^t z^{\xi}$$

and since $(v^{2^r}b^{2^{r-1}})^2 = v^{2^{r+1}}b^{2^r} = 1$, we get $(v^{2^r}b^{2^{r-1}})^t z^{\xi} = w \in W = Z(G)$.

Suppose $\zeta = 1$. In that case we have $\langle b\rangle \cap \langle b_1\rangle = F' \cap F'_1 = \langle z\rangle$, $b^{2^r} = v^{-2^{r+1}}z$, $b_1^{2^r} = z$, $(\Phi(G))' = \langle z^{\delta}\rangle$, $\delta = 0, 1$, $b^a = b^{-1}z^{\delta}$, $b_1^a = b_1z^{\delta}$. We set $u_0 = b_1^{2^{r-1}}c^{2^{s-2}}$ so that $u_0^2 = b_1^{2^r}c^{2^{s-1}} = zz = 1$ and

$$u_0^a = (b_1^{2^{r-1}}c^{2^{s-2}})^a = b_1^{2^{r-1}}c^{-2^{s-2}} = b_1^{2^{r-1}}c^{2^{s-2}}z = u_0z,$$

where we have used the facts that a inverts c and a centralizes an element of order 4 in $\langle b_1\rangle$. Hence $Z(G) = \langle z\rangle$. This implies that $a^{2^{r+2}} = z$ and so $o(a) = 2^{r+3}$. Since $c = v^{-2^{r+1}}z$ and $b_1 = v^{-2}b^{-1}z^{\epsilon}$, where $v^x = v^{-1}z^{\epsilon}$, $\epsilon = 0, 1$ (and $\epsilon = 0$ if and only if $x^2 \in \langle z\rangle$), we get

$$u_0 = (v^{-2}b^{-1}z^{\epsilon})^{2^{r-1}}(v^{-2^{r+1}}z)^{2^{s-2}} = (v^{-2^r(1+2^{s-1})}b^{-2^{r-1}})z^{2^{s-2}} = uz^{2^{s-2}},$$

where we have set $u = v^{-2^r(1+2^{s-1})}b^{-2^{r-1}}$. We see that $u^2 = 1$, $u^a = uz$ (since $u_0^a = u_0z$), and so $W = \langle u, z\rangle \cong E_4$.

Since $(b_1u^{\delta})^a = b_1z^{\delta}(uz)^{\delta} = b_1u^{\delta}$, $(b_1u^{\delta})^{2^r} = b_1^{2^r} = z$, and $C_{\langle v\rangle}(a) = \langle z\rangle$, we have $C_{G'}(a) = \langle b_1u^{\delta}\rangle$ and so

$$\langle a^4\rangle = \langle b_1u^{\delta}\rangle = \langle v^{-2}b^{-1}z^{\epsilon}u^{\delta}\rangle = \langle v^2bu^{\delta}\rangle$$

since $z^{\epsilon} \in \Phi(\langle b_1u^{\delta}\rangle)$. This gives

$$a^4 = v^2bu^{\delta}(v^2bu^{\delta})^{2i} = v^{2+4i}b^{1+2i}u^{\delta} \quad (i \text{ integer}),$$

and, therefore,

$$a^8 = v^{-4}b^{-2} = v^{4+8i}b^{2+4i},\ v^{8+8i}b^{4+4i} = 1, \quad \text{and so} \quad i + 1 \equiv 0 \,(\text{mod}\ \ 2^{r-2}).$$

We set $i = -1 + t2^{r-2}$ (t an integer) and compute:

$$1 = v^{t2^{r+1}}b^{t2^r} = (v^{2^{r+1}}b^{2^r})^t = (v^{2^{r+1}}v^{-2^{r+1}}z)^t = z^t$$

and this forces $t \equiv 0 \pmod 2$. Hence we may set $t = 2\tau$ (τ an integer) and then $i = -1 + \tau 2^{r-1}$ so that

$$a^4 = v^{2-4+\tau 2^{r+1}} b^{1-2+\tau 2^r} u^\delta = v^{-2} b^{-1} u^\delta (v^{2^{r+1}} b^{2^r})^\tau =$$

$$v^{-2} b^{-1} u^\delta (v^{2^{r+1}} v^{-2^{r+1}} z)^\tau = v^{-2} b^{-1} u^\delta z^\tau, \ \tau = 0, 1,$$

and we are done. ∎

5. Nonmetacyclic 2-groups $G = AB$ with A and B cyclic

The result of this section was also proved independently by Y. Berkovich.

THEOREM 5.1: *Let $G = AB$ be a nonmetacyclic 2-group, where the subgroups A and B are cyclic. If $\{U, V, M\}$ is the set of maximal subgroups of G, where $A < U$ and $B < V$, then U and V are metacyclic and $d(M) = 3$. Hence, M is a unique nonmetacyclic maximal subgroup of G and these groups have been completely determined in Section 4.*

Proof. Assume, for example, that U is nonmetacyclic. Then $U/\mho_2(U)$ is nonmetacyclic (Proposition 2.3) and so, in particular, $|U/\mho_2(U)| \geq 2^4$. We set $A = \langle a \rangle$ and $B = \langle b \rangle$ so that $U = \langle a \rangle \langle b^2 \rangle$, $|A : (A \cap B)| \geq 4$, and $|\langle b^2 \rangle : (A \cap B)| \geq 4$ (otherwise, U would be metacyclic). Since $a^4 \in \mho_2(U)$, $b^8 \in \mho_2(U)$, and $|U : \langle a^4, b^8 \rangle| = 2^4$ (noting that Proposition 2.12 implies that $\langle a^4, b^8 \rangle = \langle a^4 \rangle \langle b^8 \rangle$), we get $\mho_2(U) = \langle a^4 \rangle \langle b^8 \rangle$ and so $|U : \mho_2(U)| = 2^4$. We want to investigate the structure of $G/\mho_2(U)$ and so we may assume that $\mho_2(U) = \{1\}$. In that case $G = \langle a \rangle \langle b \rangle$ is a group of order 2^5 with $o(a) = 4$, $o(b) = 8$, $\langle a \rangle \cap \langle b \rangle = \{1\}$, and G has a nonmetacyclic subgroup $U = \langle a \rangle \langle b^2 \rangle$ of order 2^4 and exponent 4 which is a product of two cyclic subgroups $\langle a \rangle$ and $\langle b^2 \rangle$ of order 4.

The subgroup U is nonabelian and U is not of maximal class (otherwise, U would be metacyclic). By a result of O. Taussky (Proposition 2.4), $|U'| = 2$ and so U is minimal nonabelian (Proposition 2.9). By Proposition 2.8, $Z(U) = \Phi(U) = \langle a^2 \rangle \times \langle b^4 \rangle \cong E_4$ and $U' = \langle a^2 b^4 \rangle \leq Z(G)$ since $a^2 b^4$ is not a square in U. But $b^4 \in Z(U)$ and so $[b^4, a] = 1$ which gives $b^4 \in Z(G)$. Hence $a^2 \in Z(G)$ and we get $E_4 \cong \langle a^2, b^4 \rangle \leq Z(G)$. We have $G' \leq \Phi(G) = \langle a^2 \rangle \times \langle b^2 \rangle \cong C_2 \times C_4$ and $G' \geq U' = \langle a^2 b^4 \rangle$. We have $G' > U'$ because in case $G' = U'$, G would be minimal nonabelian and then U would be abelian, which is not the case. By the result of O. Taussky (and noting that G is not of maximal class), we get $|G/G'| \geq 8$ and so $|G'| = 4$. Hence G' is a maximal subgroup of $\Phi(G)$ and so

$G' \geq \mho_1(\Phi(G)) = \langle b^4 \rangle$. Hence $E_4 \cong G' = \langle a^2, b^4 \rangle \leq Z(G)$ and so G is of class 2. By Proposition 2.14, G' must be cyclic and this is our final contradiction.

We have proved that U and V are metacyclic. If $d(M) \leq 2$, then Proposition 2.2 implies that G is metacyclic, a contradiction. Hence $d(M) = 3$ and M is a unique nonmetacyclic maximal subgroup of G. ∎

Acknowledgement

The author thanks the referee for numerous useful comments.

References

[1] Y. Berkovich, *Groups of prime power order,* Parts I, II, (with Z. Janko), Walter de Gruyter, Berlin, 2008.

[2] Y. Berkovich, *Short proofs of some basic characterization theorems of finite p-group theory,* Glasnik Matematički **41(61)** (2006), 239–258.

[3] Y. Berkovich and Z. Janko, *Structure of finite p-groups with given subgroups,* Contemporary Mathematics **402** (2006), 13–93.

[4] B. Huppert, *Über das Produkt von paarweise vertauschbaren zyklischen Gruppen,* Mathematische Zeitschrift **58** (1953), 243–264.

[5] Z. Janko, *Finite 2-groups with no normal elementary abelian subgroups of order* 8, Journal of Algebra **246** (2001), 951–961.

[6] A. Lubotzky and A. Mann, *Powerful p-groups 1,* Journal of Algebra **105** (1987), 484–505.

[7] P. Roquette, *Darstellungen endlicher nilpotenter Gruppen,* Archiv der Mathematik **9** (1958), 241–250.